Type Systems and Functional Programming

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Part I

Introduction

Contents

Contents

Grading

- Lab: 60, ≥ 30
- Exam: 40, ≥ 20
- Final grade ≥ 50

Course objectives

- Study the characteristics of functional programming, such as *lazy evaluation* and *type systems* of different strengths

- Learn advanced mechanisms of the Haskell language, which are impossible or difficult to simulate in other languages

- Apply this apparatus to model practical problems

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- Study the characteristics of functional programming, such as *lazy evaluation* and *type systems* of different strengths
- Learn advanced mechanisms of the Haskell language, which are impossible or difficult to simulate in other languages
- Apply this apparatus to model practical problems, e.g. program synthesis, lazy search, probability spaces

Main lab outcome

An evaluator for a functional language,
equipped with a type synthesizer

Contents

Functional programming features

- Mathematical functions, as value transformers

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- Functions as first-class values

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- Mathematical functions, as value transformers
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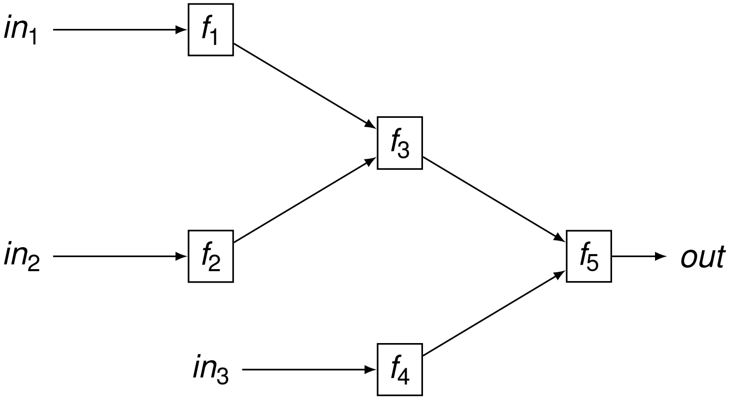
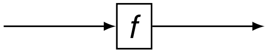
- No side effects or state

Functional flow



Stateless computation

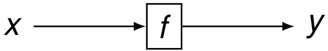
Output dependent on input exlcusively:



t_0

Stateless computation

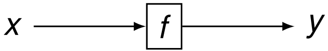
Output dependent on input exlcusively:



t_1

Stateless computation

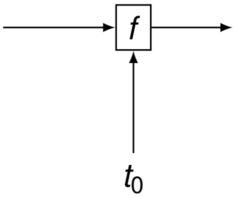
Output dependent on input exlcusively:



t_2

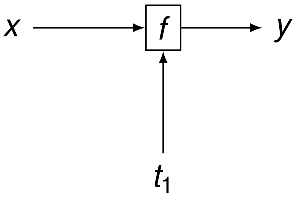
Stateful computation

Output dependent on input and time:



Stateful computation

Output dependent on input and time:



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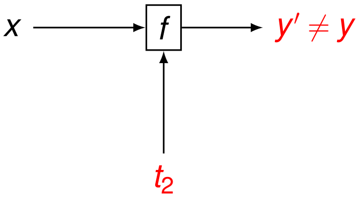
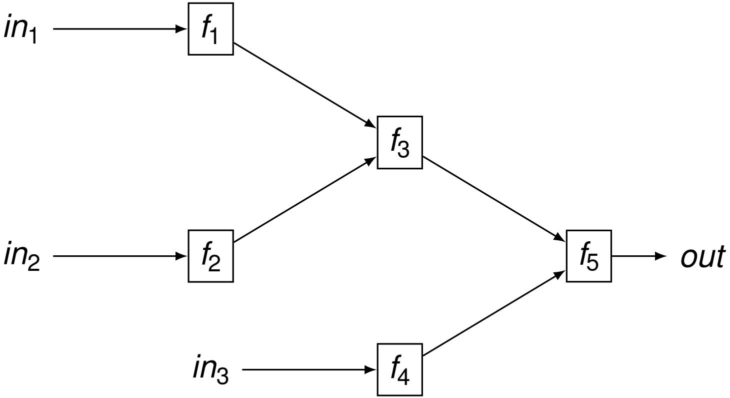


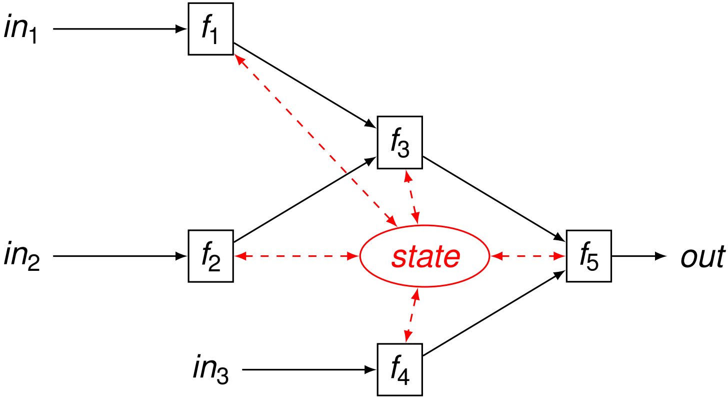
Functional flow

Pure



Functional flow

Impure



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- Lazy evaluation

Why functional programming?

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- Type systems and logic, inextricably linked
- Automatic program verification and synthesis

Part II

Untyped Lambda Calculus

Contents

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Untyped lambda calculus

- Model of computation — Alonzo Church, 1932

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Applications

- Theoretical basis of numerous languages:
  - LISP
  - Scheme
  - Haskell
  - ML
  - F#
  - Clean
  - Clojure
  - Scala
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  - LISP
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  - Clojure
  - Scala
  - Erlang
- Formal program verification, due to its simple execution model

Contents

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- Variable: a variable x is a λ-expression
- Function: if x is a variable and E is a λ-expression, then $\lambda x.E$ is a λ-expression, which stands for an anonymous, unary function, with the formal parameter x and the body E
- Application: if E and A are λ-expressions, then $(E\ A)$ is a λ-expression, which stands for the application of the expression E onto the actual argument A.

λ-expressions
Examples

Example 4.2 (λ-expressions).

- $x \rightarrow$ variable x

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λ-expressions

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- $\lambda x.\lambda y.x$: a function with another function as body!
- $(\lambda x.x\ \ y)$: the application of the identity function onto the actual argument y
- $(\lambda x.(x\ \ x)\ \ \lambda x.x)$

Intuition on application evaluation

$$(\lambda\ x\ .\quad x \qquad y\ )$$

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Intuition on application evaluation

$$(\lambda x . x \; y)$$

Intuition on application evaluation

$$(\lambda x . x \; y) \to y$$

Variable occurrences

Definitions

Definition 4.3 (Bound occurrence).

An occurrence x_n of a variable x is bound in the expression E iff:

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Definition 4.4 (Free occurrence).

A variable occurrence is free in an expression iff it is not bound in that expression.

Bound/ free occurrence w.r.t. a given expression!

Example 4.5 (Bound and free variables).

In the expression $E = (\lambda x.x\ \ x)$, we emphasize the occurrences of x:

$$E = (\lambda x_1.\underbrace{x_2}_{F}\ \ x_3).$$

Variable occurrences

Examples

Example 4.5 (Bound and free variables).

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- x_3 free in E

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Example 4.6 (Bound and free variables).

In the expression $E = (\lambda x.\lambda z.(z\ \ x)\ \ (z\ \ y))$, we emphasize the occurrences of x, y, z:

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Variable occurrences

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- x_2 free in F!
- x free in E and F

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- z free in E, but bound in F

Free and bound variables

Free variables

- $FV(x) =$

Free and bound variables

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- $FV(x) = \{x\}$
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Bound variables

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Bound variables

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Closed expressions

Definition 4.9 (Closed expression).

An expression that does not contain any free variables.

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Example 4.10 (Closed expressions).

- $(\lambda x.x \ \ \lambda x.\lambda y.x)$

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Definition 4.9 (Closed expression).

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Example 4.10 (Closed expressions).

- $(\lambda x.x \ \lambda x.\lambda y.x)$: closed
- $(\lambda x.x \ a)$

Closed expressions

Definition 4.9 (Closed expression).

An expression that does not contain any free variables.

Example 4.10 (Closed expressions).

- $(\lambda x.x \;\; \lambda x.\lambda y.x)$: closed
- $(\lambda x.x \;\; a)$: open, since a is free

Remarks:

- Free variables may stand for other λ-expressions, as in $\lambda x.((+ \;\; x) \;\; 1)$.
- Before evaluation, an expression must be brought to the closed form.
- The substitution process must terminate.

Contents

Definition 5.1 (β-reduction).
The evaluation of the application $(\lambda x.E\ A)$, by substituting every free occurrence of the formal argument, x, in the function body, E, with the actual argument, A:
$(\lambda x.E\ A) \rightarrow_\beta E_{[A/x]}$.

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Definition 5.2 (β-redex).

The application $(\lambda x.E\ A)$.

β-reduction

Examples

Example 5.3 (β-reduction).

- $(\lambda x.x\ \ y)$

Example 5.3 (β-reduction).

- $(\lambda x.x\ \ y) \rightarrow_{\beta} x_{[y/x]}$

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- $(\lambda x.x \ \ y) \rightarrow_\beta x_{[y/x]} \rightarrow y$
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Example 5.3 (β-reduction).

- $(\lambda x.x\ \ y) \rightarrow_\beta x_{[y/x]} \rightarrow y$
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β-reduction

Examples

Example 5.3 (β-reduction).

- $(\lambda x.x \; y) \rightarrow_\beta x_{[y/x]} \rightarrow y$
- $(\lambda x.\lambda x.x \; y) \rightarrow_\beta \lambda x.x_{[y/x]} \rightarrow \lambda x.x$
- $(\lambda x.\lambda y.x \; y)$

β-reduction

Examples

Example 5.3 (β-reduction).

- $(\lambda x.x\ \ y) \to_\beta x_{[y/x]} \to y$
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  Wrong! The free variable y becomes bound, changing its meaning!

β-reduction
Collisions

- Problem: within the expression $(\lambda x.E\ \ A)$:

β-reduction
Collisions

- Problem: within the expression $(\lambda x.E\ A)$:
  - $FV(A) \cap BV(E) = \emptyset \Rightarrow$ correct reduction always

β-reduction

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Example 5.4 (Bound variable renaming).

$(\lambda x.\lambda y.x\ \ y)$

β-reduction

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Example 5.4 (Bound variable renaming).

$(\lambda x.\lambda y.x\ \ y) \rightarrow (\lambda x.\lambda z.x\ \ y)$

β-reduction
Collisions

- Problem: within the expression $(\lambda x.E\ \ A)$:
  - $FV(A) \cap BV(E) = \emptyset \Rightarrow$ correct reduction always
  - $FV(A) \cap BV(E) \neq \emptyset \Rightarrow$ potentially wrong reduction
- Solution: rename the bound variables in E, that are free in A

Example 5.4 (Bound variable renaming).

$(\lambda x.\lambda y.x\ \ y) \rightarrow (\lambda x.\lambda z.x\ \ y) \rightarrow_{\beta} \lambda z.x_{[y/x]}$

β-reduction

Collisions

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Example 5.4 (Bound variable renaming).

$(\lambda x.\lambda y.x\ \ y) \to (\lambda x.\lambda z.x\ \ y) \to_\beta \lambda z.x_{[y/x]} \to \lambda z.y$

α-conversion

Definition 5.5 (α-conversion).

Systematic relabeling of bound variables in a function: $\lambda x.E \rightarrow_{\alpha} \lambda y.E_{[y/x]}$. Two conditions must be met.

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- $\lambda x.y \rightarrow_\alpha \lambda y.y_{[y/x]}$

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- $\lambda x.y \rightarrow_\alpha \lambda y.y_{[y/x]} \rightarrow \lambda y.y$: Wrong!
- $\lambda x.\lambda y.x$

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Conditions:

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Conditions:

- y is not free in E

Definition 5.5 (α-conversion).

Systematic relabeling of bound variables in a function: $\lambda x.E \rightarrow_\alpha \lambda y.E_{[y/x]}$. Two conditions must be met.

Example 5.6 (α-conversion).

- $\lambda x.y \rightarrow_\alpha \lambda y.y_{[y/x]} \rightarrow \lambda y.y$: Wrong!
- $\lambda x.\lambda y.x \rightarrow_\alpha \lambda y.\lambda y.x_{[y/x]} \rightarrow \lambda y.\lambda y.y$: Wrong!

Conditions:

- y is not free in E
- a free occurrence in E stays free in $E_{[y/x]}$

α-conversion

Examples

Example 5.7 (α-conversion).

- $\lambda x.(x\ \ y) \rightarrow_\alpha \lambda z.(z\ \ y)$

α-conversion

Examples

Example 5.7 (α-conversion).

- $\lambda x.(x\ \ y) \rightarrow_{\alpha} \lambda z.(z\ \ y)$: Correct!
- $\lambda x.\lambda x.(x\ \ y) \rightarrow_{\alpha} \lambda y.\lambda x.(x\ \ y)$

α-conversion

Examples

Example 5.7 (α-conversion).

- $\lambda x.(x\ \ y) \rightarrow_\alpha \lambda z.(z\ \ y)$: Correct!
- $\lambda x.\lambda x.(x\ \ y) \rightarrow_\alpha \lambda y.\lambda x.(x\ \ y)$: Wrong!
  y is free in $\lambda x.(x\ \ y)$.
- $\lambda x.\lambda y.(y\ \ x) \rightarrow_\alpha \lambda y.\lambda y.(y\ \ y)$

α-conversion

Examples

Example 5.7 (α-conversion).

- $\lambda x.(x\ \ y) \to_\alpha \lambda z.(z\ \ y)$: Correct!
- $\lambda x.\lambda x.(x\ \ y) \to_\alpha \lambda y.\lambda x.(x\ \ y)$: Wrong!
  y is free in $\lambda x.(x\ \ y)$.
- $\lambda x.\lambda y.(y\ \ x) \to_\alpha \lambda y.\lambda y.(y\ \ y)$: Wrong!
  The free occurrence of x in $\lambda y.(y\ \ x)$ becomes bound, after substitution, in $\lambda y.(y\ \ y)$.
- $\lambda x.\lambda y.(y\ \ y) \to_\alpha \lambda y.\lambda y.(y\ \ y)$

α-conversion

Examples

Example 5.7 (α-conversion).

- $\lambda x.(x \; y) \rightarrow_\alpha \lambda z.(z \; y)$: Correct!
- $\lambda x.\lambda x.(x \; y) \rightarrow_\alpha \lambda y.\lambda x.(x \; y)$: Wrong!
  y is free in $\lambda x.(x \; y)$.
- $\lambda x.\lambda y.(y \; x) \rightarrow_\alpha \lambda y.\lambda y.(y \; y)$: Wrong!
  The free occurrence of x in $\lambda y.(y \; x)$ becomes bound, after substitution, in $\lambda y.(y \; y)$.
- $\lambda x.\lambda y.(y \; y) \rightarrow_\alpha \lambda y.\lambda y.(y \; y)$: Correct!

Definition 5.8 (Reduction step).

A sequence made of a possible α-conversion, followed by a β-reduction, such that the second produces no collisions: $E_1 \rightarrow E_2 \equiv E_1 \rightarrow_\alpha E_3 \rightarrow_\beta E_2$.

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Definition 5.9 (Reduction sequence).

A string of zero or more reduction steps: $E_1 \rightarrow^* E_2$. It is an element of the reflexive transitive closure of relation $\rightarrow$.

Example 5.10 (Reduction).

- $((\lambda x.\lambda y.(y\ \ x)\ \ y)\ \ \lambda x.x)$

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- $((\lambda x.\lambda y.(y\ \ x)\ \ y)\ \ \lambda x.x)$
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  $\rightarrow (\lambda x.x\ \ y)$
  $\rightarrow y$

- $((\lambda x.\lambda y.(y\ \ x)\ \ y)\ \ \lambda x.x) \rightarrow^* y$

Reduction

Properties

- Reduction step = reduction sequence:

$$E_1 \rightarrow E_2 \Rightarrow E_1 \rightarrow^* E_2$$

- Reflexivity:

$$E \rightarrow^* E$$

- Transitivity:

$$E_1 \rightarrow^* E_2 \wedge E_2 \rightarrow^* E_3 \Rightarrow E_1 \rightarrow^* E_3$$

Contents

Questions

1. When does the computation terminate?
   Does it always?

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Normal forms

Definition 6.1 (Normal form).

The form of an expression that cannot be reduced i.e., that contains no β-redexes.

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Example 6.3 (Normal forms).

$(\lambda x.\lambda y.(x\ \ y)\ \ \lambda x.x) \rightarrow_{\text{FNF}} \lambda y.(\lambda x.x\ \ y) \rightarrow_{\text{NF}} \lambda y.y$

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$(\lambda x.\lambda y.(x\ \ y)\ \ \lambda x.x) \rightarrow_{\text{FNF}} \lambda y.(\lambda x.x\ \ y) \rightarrow_{\text{NF}} \lambda y.y$

FNF is used in programming, where the function body is evaluated only when the function is effectively applied.

Reduction termination (reducibility)

Example 6.4.

$\Omega \equiv (\lambda x.(x\ \ x)\ \ \lambda x.(x\ \ x))$

Reduction termination (reducibility)

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$\Omega \equiv (\lambda x.(x\ \ x)\ \ \lambda x.(x\ \ x)) \rightarrow (\lambda x.(x\ \ x)\ \ \lambda x.(x\ \ x)) \rightarrow^{*} \dots$

Ω does not have a terminating reduction sequence.

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Definition 6.5 (Reducible expression).

An expression that has a terminating reduction sequence.

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$\Omega \equiv (\lambda x.(x\ x)\ \lambda x.(x\ x)) \rightarrow (\lambda x.(x\ x)\ \lambda x.(x\ x)) \rightarrow^* \ldots$

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An expression that has a terminating reduction sequence.

Ω is irreducible.

Questions

1. When does the computation terminate?
   Does it always?

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Example 6.6 (Reduction sequences).

$$E = (\lambda x.y \ \Omega)$$

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- $\xrightarrow{1} y$

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- $\xrightarrow{2^n 1}{}^{*}\ y,\ n \geq 0$

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- $\xrightarrow{2^\infty}{}^{*}\ \dots$

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- $\ldots$

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- E has a nonterminating reduction sequence, but still has a normal form, y. E is reducible, Ω is not.

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- $\xrightarrow{2} E \xrightarrow{2} E \xrightarrow{1} y$
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- $\xrightarrow{2^\infty}{}^{*}\ \ldots$

- E has a nonterminating reduction sequence, but still has a normal form, y. E is reducible, Ω is not.
- The length of terminating reduction sequences is unbounded.

Questions

1. When does the computation terminate?
   Does it always?
   - NO
2. Does the answer depend on the reduction sequence?
3. If the computation terminates for distinct reduction sequences, do we always get the same result?
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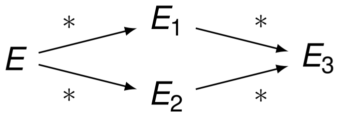
Normal form uniqueness

Results

Theorem 6.7 (Church-Rosser / diamond).
If $E \to^* E_1$ and $E \to^* E_2$, then there is an E_3 such that $E_1 \to^* E_3$ and $E_2 \to^* E_3$.

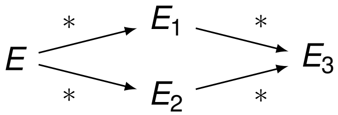


Normal form uniqueness

Results

Theorem 6.7 (Church-Rosser / diamond).

If $E \to^* E_1$ and $E \to^* E_2$, then there is *an E_3 such that $E_1 \to^* E_3$ and $E_2 \to^* E_3$.*



Corollary 6.8 (Normal form uniqueness).

If an expression is reducible, its normal form is unique. It corresponds to the value of that expression.

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ \ y)\ \ (\lambda x.x\ \ y))$$

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ y)\ (\lambda x.x\ y))$$

- $\rightarrow \lambda z.((\lambda x.x\ y)\ z) \rightarrow \lambda z.(y\ z)$

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ \ y)\ \ (\lambda x.x\ \ y))$$

- $\rightarrow \lambda z.((\lambda x.x\ \ y)\ \ z) \rightarrow \lambda z.(y\ \ z)$
- $\rightarrow (\lambda x.\lambda y.(x\ \ y)\ \ y) \rightarrow \lambda w.(y\ \ w)$

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ \ y)\ \ (\lambda x.x\ \ y))$$

- $\rightarrow \lambda z.((\lambda x.x\ \ y)\ \ z) \rightarrow \lambda z.(y\ \ z) \rightarrow_\alpha \lambda a.(y\ \ a)$
- $\rightarrow (\lambda x.\lambda y.(x\ \ y)\ \ y) \rightarrow \lambda w.(y\ \ w) \rightarrow_\alpha \lambda a.(y\ \ a)$

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ y)\ (\lambda x.x\ y))$$

- $\rightarrow \lambda z.((\lambda x.x\ y)\ z) \rightarrow \lambda z.(y\ z) \rightarrow_\alpha \lambda a.(y\ a)$
- $\rightarrow (\lambda x.\lambda y.(x\ y)\ y) \rightarrow \lambda w.(y\ w) \rightarrow_\alpha \lambda a.(y\ a)$

- Normal form: class of expressions, equivalent under systematic relabeling

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ y)\ (\lambda x.x\ y))$$

- $\rightarrow \lambda z.((\lambda x.x\ y)\ z) \rightarrow \lambda z.(y\ z) \rightarrow_\alpha \lambda a.(y\ a)$
- $\rightarrow (\lambda x.\lambda y.(x\ y)\ y) \rightarrow \lambda w.(y\ w) \rightarrow_\alpha \lambda a.(y\ a)$

- Normal form: class of expressions, equivalent under systematic relabeling
- Value: distinguished member of this class

Structural equivalence

Definition 6.10 (Structural equivalence).
Two expressions are structurally equivalent iff they both reduce to the same expression.

Example 6.11 (Structural equivalence).
$\lambda z.((\lambda x.x \ \ y) \ \ z)$ and $(\lambda x.\lambda y.(x \ \ y) \ \ y)$ in Example 6.9.

Computational equivalence

Definition 6.12 (Computational equivalence).

Two expressions are computationally equivalent iff they the behave in the same way when applied onto the same arguments.

Example 6.13 (Computational equivalence).

$$E_1 = \lambda y.\lambda x.(y\ \ x)$$
$$E_2 = \lambda x.x$$

- $((E_1\ \ a)\ \ b) \rightarrow^* (a\ \ b)$
- $((E_2\ \ a)\ \ b) \rightarrow^* (a\ \ b)$

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Example 6.13 (Computational equivalence).

$$E_1 = \lambda y.\lambda x.(y\ \ x)$$
$$E_2 = \lambda x.x$$

- $((E_1\ \ a)\ \ b) \rightarrow^* (a\ \ b)$
- $((E_2\ \ a)\ \ b) \rightarrow^* (a\ \ b)$
- $E_1 \not\rightarrow^* E_2$ and $E_2 \not\rightarrow^* E_1$ (not structurally equivalent)

Questions

1. When does the computation terminate?
   Does it always?
   - NO
2. Does the answer depend on the reduction sequence?
   - YES
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Reduction order

Definitions and examples

Definition 6.14 (Left-to-right reduction step).

The reduction of the outermost leftmost β-redex.

Example 6.15 (Left-to-right reduction).

$$(\underline{(\lambda x.x\ \lambda x.y)}\ (\lambda x.(x\ x)\ \lambda x.(x\ x))) \to \underline{(\lambda x.y\ \Omega)} \to y$$

Reduction order

Definitions and examples

Definition 6.14 (Left-to-right reduction step).
The reduction of the outermost leftmost β-redex.

Example 6.15 (Left-to-right reduction).
$(\underline{(\lambda x.x\ \lambda x.y)}\ (\lambda x.(x\ x)\ \lambda x.(x\ x))) \rightarrow \underline{(\lambda x.y\ \Omega)} \rightarrow y$

Definition 6.16 (Right-to-left reduction step).
The reduction of the innermost rightmost β-redex.

Example 6.17 (Right-to-left reduction).
$((\lambda x.x\ \lambda x.y)\ \underline{(\lambda x.(x\ x)\ \lambda x.(x\ x))}) \rightarrow (\lambda x.y\ \underline{\Omega}) \rightarrow \ldots$

Reduction order

Which one is better?

Theorem 6.18 (Normalization).

If an expression is reducible, its left-to-right reduction terminates.

The theorem does not guarantee the termination for any expression, but only for reducible ones!

Questions

1. When does the computation terminate?
   Does it always?
   - NO
2. Does the answer depend on the reduction sequence?
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   - YES
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   - YES
4. If the result is unique, how do we safely obtain it?
   - Left-to-right reduction

Contents

Evaluation order

Definition 7.1 (Applicative-order evaluation).

Corresponds to right-to-left reduction. Function arguments are evaluated before the function is applied.

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A function that uses applicative-order evaluation.

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Definition 7.3 (Normal-order evaluation).
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Evaluation order

Definition 7.1 (Applicative-order evaluation).
Corresponds to right-to-left reduction. Function arguments are evaluated before the function is applied.

Definition 7.2 (Strict function).
A function that uses applicative-order evaluation.

Definition 7.3 (Normal-order evaluation).
Corresponds to left-to-right reduction. Function arguments are evaluated when needed.

Definition 7.4 (Non-strict function).
A function that uses normal-order evaluation.

In practice I

Applicative-order evaluation employed in most programming languages, due to efficiency — one-time evaluation of arguments: C, Java, Scheme, PHP, etc.

Example 7.5 (Applicative-order evaluation in Scheme).

$\texttt{((}\lambda\texttt{ (x) (+ x x)) }\underline{\texttt{(+ 2 3)}}\texttt{)}$

$\rightarrow \underline{\texttt{((}\lambda\texttt{ (x) (+ x x)) 5)}}$

$\rightarrow \underline{\texttt{(+ 5 5)}}$

$\rightarrow \texttt{10}$

In practice II

Lazy evaluation (a kind of normal-order evaluation) in Haskell: on-demand evaluation of arguments, allowing for interesting constructions

Example 7.6 (Lazy evaluation in Haskell).

```
((\x -> x + x) (2 + 3))
→ (2 + 3) + (2 + 3)
→ 5 + 5
→ 10
```

Need for non-strict functions, even in applicative languages: `if`, `and`, `or`, etc.

Summary

- Lambda calculus: model of computation, underpinned by functions and textual substitution
- Bound/free variables and variable occurrences w.r.t. an expression
- β-reduction, α-conversion, reduction step, reduction sequence, reduction order, normal forms
- Left-to-right reduction (normal-order evaluation): always terminates for reducible expressions
- Right-to-left reduction (applicative-order evaluation): more efficient but no guarantee on termination even for reducible expressions!

Part III

Lambda Calculus
as a Programming Language

Contents

Contents

Purpose

- Proving the expressive power of lambda calculus

Purpose

- Proving the expressive power of lambda calculus
- Hypothetical λ-machine

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- Machine code: λ-expressions — the λ_0 language

Purpose

- Proving the expressive power of lambda calculus
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- Machine code: λ-expressions — the λ_0 language
- Instead of
  - bits
  - bit operations,

  we have

Purpose

- Proving the expressive power of lambda calculus
- Hypothetical λ-machine
- Machine code: λ-expressions — the λ_0 language
- Instead of
  - bits
  - bit operations,

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  - structured strings of symbols

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- Machine code: λ-expressions — the λ_0 language
- Instead of
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  - reduction — textual substitution

λ_0 features

- Instructions:

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  - top-level variable bindings: *variable* $\equiv_{\text{def}}$ *expression*
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- Expressions brought to the closed form,
  prior to evaluation
- Normal-order evaluation
- Functional normal form (see Definition 6.2)
- No predefined types!

Shorthands

- $\lambda x_1.\lambda x_2.\lambda \ldots .\lambda x_n.E \to \lambda x_1 x_2 \ldots x_n.E$

- $((\ldots((E\ A_1)\ A_2)\ \ldots)\ A_n) \to (E\ A_1\ A_2\ \ldots\ A_n)$

Purpose of types

- Way of expressing the programmer's intent

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- Formal verification

No types

How are objects represented?

- A number, list or tree potentially designated by the same value e.g.,

$$\text{number } 3 \rightarrow \lambda x.\lambda y.x \leftarrow \text{list } ((\,)\ (\,)\ (\,))$$

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How are objects represented?

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$$\text{number } 3 \rightarrow \lambda x.\lambda y.x \leftarrow \text{list } (()\ ()\ ())$$

- Both values and operators represented by functions — context-dependent meaning

$$\text{number } 3 \rightarrow \lambda x.\lambda y.x \leftarrow \text{operator } \textit{car}$$

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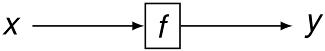
  number 3 $\rightarrow \lambda x.\lambda y.x \leftarrow$ operator *car*

- Value applicable onto another value, as an operator!



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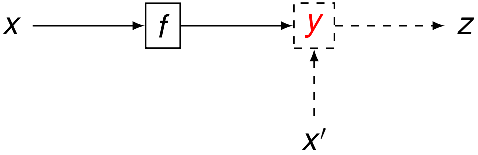

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- Inability of the λ machine to
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  - ensure their correctness

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- Erroneus constructs accepted without warning, but computation ended with
  - values with no meaning or
  - expressions that are neither values, nor reducible e.g., $(x\ \ x)$

No types
Consequences

- Enhanced representational flexibility

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- Useful when the uniform representation of objects, as lists de symbols, is convenient
- Increased error-proneness
- Program instability
- Difficulty of verification and maintenance

So...

- How do we employ the λ_0 language in everyday programming?

So...

- How do we employ the λ_0 language in everyday programming?

- How do we represent usual values — numbers, booleans, lists, etc. — and their corresponding operators?

Contents

Definition

Definition 9.1 (Abstract data type, ADT).
Mathematical model of a set of values and their corresponding operations.

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Natural, *Bool*, *List*, *Set*, *Stack*, *Tree*, ... λ-expression!

Definition

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Natural, *Bool*, *List*, *Set*, *Stack*, *Tree*, ... λ-expression!

Components:

- base constructors: how are values built
- operators: what can be done with these values
- axioms: how

The *Natural* ADT

Base constructors and operators

- Base constructors:

- Operators:

The *Natural* ADT

Base constructors and operators

- Base constructors:
  - $\mathit{zero} : \to \mathit{Natural}$
- Operators:

The *Natural* ADT

Base constructors and operators

- Base constructors:
  - $zero : \to Natural$
  - $succ : Natural \to Natural$
- Operators:

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The *Natural* ADT

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  - $zero : \rightarrow Natural$
  - $succ : Natural \rightarrow Natural$
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  - $zero? : Natural \rightarrow Bool$
  - $pred : Natural \setminus \{zero\} \rightarrow Natural$

The *Natural* ADT

Base constructors and operators

- Base constructors:
  - $zero : \rightarrow Natural$
  - $succ : Natural \rightarrow Natural$

- Operators:
  - $zero? : Natural \rightarrow Bool$
  - $pred : Natural \setminus \{zero\} \rightarrow Natural$
  - $add : Natural^2 \rightarrow Natural$

The *Natural* ADT

Axioms

- *zero*?

- *pred*

- *add*

The *Natural* ADT

Axioms

- *zero*?
  - $(zero?\ zero) = T$
- *pred*
- *add*

The *Natural* ADT

Axioms

- *zero?*
  - $(zero?\ zero) = T$
  - $(zero?\ (succ\ n)) = F$
- *pred*
- *add*

The *Natural* ADT

Axioms

- *zero*?
  - $(zero?\ zero) = T$
  - $(zero?\ (succ\ n)) = F$
- *pred*
  - $(pred\ (succ\ n)) = n$
- *add*

The *Natural* ADT

Axioms

- *zero?*
  - $(zero?\ \ zero) = T$
  - $(zero?\ \ (succ\ \ n)) = F$
- *pred*
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- *add*
  - $(add\ \ zero\ \ n) = n$

The *Natural* ADT

Axioms

- *zero*?
  - $(zero?\ zero) = T$
  - $(zero?\ (succ\ n)) = F$
- *pred*
  - $(pred\ (succ\ n)) = n$
- *add*
  - $(add\ zero\ n) = n$
  - $(add\ (succ\ m)\ n) = (succ\ (add\ m\ n))$

Providing axioms

- One axiom for each (operator, base constructor) pair

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Providing axioms

- One axiom for each (operator, base constructor) pair
- More — useless
- Less — insufficient for completely specifying the operators

From ADTs to functional programming

Exemple

- **Axiome**:
  - $add(zero, n) = n$
  - $add(succ(m), n) = succ(add(m, n))$
- **Scheme**:

```scheme
(define add
  (lambda (m n)
    (if (zero? m) n
        (+ 1 (add (- m 1) n)))))
```

- **Haskell**:

```haskell
add 0 n = n
add (m + 1) n = 1 + (add m n)
```

From ADTs to functional programming
Discussion

- Proving ADT correctness
  — structural induction

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- Proving properties of λ-expressions, seen as values of an ADT with 3 base constructors!

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  — natural instrument, inherited from axioms

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Discussion

- Proving ADT correctness
  — structural induction
- Proving properties of λ-expressions, seen as values of an ADT with 3 base constructors!
- Functional programming
  — reflection of mathematical specifications
- Recursion
  — natural instrument, inherited from axioms
- Applying formal methods on the recursive code, taking advantage of the lack of side effects

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The *Bool* ADT

Base contrsuctors and operators

- Base constructors:
- Operators:

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- Base constructors:
  - $T : \to Bool$
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- Base constructors:
  - $T : \to Bool$
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  - $not : Bool \to Bool$

The *Bool* ADT

Base contrsuctors and operators

- Base constructors:
  - $T : \to Bool$
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The *Bool* ADT

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  - $or : Bool^2 \to Bool$
  - $if : Bool \times T \times T \to T$

The *Bool* ADT

Axioms

- *not*
- *and*
- *or*
- *if*

The *Bool* ADT

Axioms

- *not*
  - $(not \;\; T) = F$
- *and*
- *or*
- *if*

The *Bool* ADT

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  - $(not \;\; T) = F$
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The *Bool* ADT

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The *Bool* ADT

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The *Bool* ADT

Base constructor implementation

- Intuition: selecting one of the two values, *true* or *false*
- $T \equiv_{\text{def}} \lambda xy.x$
- $F \equiv_{\text{def}} \lambda xy.y$
- Selector-like behavior:
  - $(T\ a\ b) \rightarrow (\lambda xy.x\ a\ b) \rightarrow a$
  - $(F\ a\ b) \rightarrow (\lambda xy.y\ a\ b) \rightarrow b$

The *Bool* ADT

Operator implementation

- $not \equiv_{\text{def}}$
  - $(not\ \ T) \qquad \to F$
  - $(not\ \ F) \qquad \to T$
- $and \equiv_{\text{def}}$
  - $(and\ \ T\ \ a) \qquad \to a$
  - $(and\ \ F\ \ a) \qquad \to F$
- $or \equiv_{\text{def}}$
  - $(or\ \ T\ \ a) \qquad \to T$
  - $(or\ \ F\ \ a) \qquad \to a$
- $if \equiv_{\text{def}}$
  - $(if\ \ T\ \ a\ \ b) \qquad \to a$
  - $(if\ \ F\ \ a\ \ b) \qquad \to b$

The *Bool* ADT

Operator implementation

- $not \equiv_{\text{def}} \lambda x.(x\ \ F\ \ T)$
  - $(not\ \ T) \rightarrow (\lambda x.(x\ \ F\ \ T)\ \ T) \rightarrow (T\ \ F\ \ T) \rightarrow F$
  - $(not\ \ F) \rightarrow (\lambda x.(x\ \ F\ \ T)\ \ F) \rightarrow (F\ \ F\ \ T) \rightarrow T$
- $and \equiv_{\text{def}}$
  - $(and\ \ T\ \ a) \qquad \rightarrow a$
  - $(and\ \ F\ \ a) \qquad \rightarrow F$
- $or \equiv_{\text{def}}$
  - $(or\ \ T\ \ a) \qquad \rightarrow T$
  - $(or\ \ F\ \ a) \qquad \rightarrow a$
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  - $(if\ \ T\ \ a\ \ b) \qquad \rightarrow a$
  - $(if\ \ F\ \ a\ \ b) \qquad \rightarrow b$

The *Bool* ADT

Operator implementation

- $not \equiv_{\text{def}} \lambda x.(x\ F\ T)$
  - $(not\ T) \rightarrow (\lambda x.(x\ F\ T)\ T) \rightarrow (T\ F\ T) \rightarrow F$
  - $(not\ F) \rightarrow (\lambda x.(x\ F\ T)\ F) \rightarrow (F\ F\ T) \rightarrow T$
- $and \equiv_{\text{def}} \lambda xy.(x\ y\ F)$
  - $(and\ T\ a) \rightarrow (\lambda xy.(x\ y\ F)\ T\ a) \rightarrow (T\ a\ F) \rightarrow a$
  - $(and\ F\ a) \rightarrow (\lambda xy.(x\ y\ F)\ F\ a) \rightarrow (F\ a\ F) \rightarrow F$
- $or \equiv_{\text{def}}$
  - $(or\ T\ a) \qquad \rightarrow T$
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- $or \equiv_{\text{def}} \lambda xy.(x\ T\ y)$
  - $(or\ T\ a) \rightarrow (\lambda xy.(x\ T\ y)\ T\ a) \rightarrow (T\ T\ a) \rightarrow T$
  - $(or\ F\ a) \rightarrow (\lambda xy.(x\ T\ y)\ F\ a) \rightarrow (F\ T\ a) \rightarrow a$
- $if \equiv_{\text{def}}$
  - $(if\ T\ a\ b) \qquad \rightarrow a$
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The *Bool* ADT

Operator implementation

- $not \equiv_{\text{def}} \lambda x.(x\ \ F\ \ T)$
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  - $(not\ \ F) \rightarrow (\lambda x.(x\ \ F\ \ T)\ \ F) \rightarrow (F\ \ F\ \ T) \rightarrow T$
- $and \equiv_{\text{def}} \lambda xy.(x\ \ y\ \ F)$
  - $(and\ \ T\ \ a) \rightarrow (\lambda xy.(x\ \ y\ \ F)\ \ T\ \ a) \rightarrow (T\ \ a\ \ F) \rightarrow a$
  - $(and\ \ F\ \ a) \rightarrow (\lambda xy.(x\ \ y\ \ F)\ \ F\ \ a) \rightarrow (F\ \ a\ \ F) \rightarrow F$
- $or \equiv_{\text{def}} \lambda xy.(x\ \ T\ \ y)$
  - $(or\ \ T\ \ a) \rightarrow (\lambda xy.(x\ \ T\ \ y)\ \ T\ \ a) \rightarrow (T\ \ T\ \ a) \rightarrow T$
  - $(or\ \ F\ \ a) \rightarrow (\lambda xy.(x\ \ T\ \ y)\ \ F\ \ a) \rightarrow (F\ \ T\ \ a) \rightarrow a$
- $if \equiv_{\text{def}} \lambda cte.(c\ \ t\ \ e)$ non-strict!
  - $(if\ \ T\ \ a\ \ b) \rightarrow (\lambda cte.(c\ \ t\ \ e)\ \ T\ \ a\ \ b) \rightarrow (T\ \ a\ \ b) \rightarrow a$
  - $(if\ \ F\ \ a\ \ b) \rightarrow (\lambda cte.(c\ \ t\ \ e)\ \ F\ \ a\ \ b) \rightarrow (F\ \ a\ \ b) \rightarrow b$

The *Pair* ADT

Specification

- Base constructors:

- Operators:

- Axioms:

The *Pair* ADT

Specification

- Base constructors:
  - $pair : A \times B \to Pair$
- Operators:
- Axioms:

The *Pair* ADT

Specification

- Base constructors:
  - $pair : A \times B \rightarrow Pair$
- Operators:
  - $fst : Pair \rightarrow A$
- Axioms:

The *Pair* ADT

Specification

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The *Pair* ADT

Implementation

- Intuition: a pair = a function that expects a selector, in order to apply it onto its components

- *pair* $\equiv_{\text{def}}$
  - ($pair\ \ a\ \ b$)

- *fst* $\equiv_{\text{def}}$
  - ($fst\ \ (pair\ \ a\ \ b)$)

    $\rightarrow a$

- *snd* $\equiv_{\text{def}}$
  - ($snd\ \ (pair\ \ a\ \ b)$)

    $\rightarrow b$

The *Pair* ADT
Implementation

- Intuition: a pair = a function that expects a selector, in order to apply it onto its components

- $pair \equiv_{\text{def}} \lambda xys.(s\ x\ y)$
  - $(pair\ a\ b) \rightarrow (\lambda xys.(s\ x\ y)\ a\ b) \rightarrow \lambda s.(s\ a\ b)$

- $fst \equiv_{\text{def}}$
  - $(fst\ (pair\ a\ b))$

    $\rightarrow a$

- $snd \equiv_{\text{def}}$
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- Intuition: a pair = a function that expects a selector, in order to apply it onto its components

- $pair \equiv_{\text{def}} \lambda xys.(s\ x\ y)$
  - $(pair\ a\ b) \rightarrow (\lambda xys.(s\ x\ y)\ a\ b) \rightarrow \lambda s.(s\ a\ b)$

- $fst \equiv_{\text{def}} \lambda p.(p\ T)$
  - $(fst\ (pair\ a\ b)) \rightarrow (\lambda p.(p\ T)\ \lambda s.(s\ a\ b)) \rightarrow (\lambda s.(s\ a\ b)\ T) \rightarrow (T\ a\ b) \rightarrow a$

- $snd \equiv_{\text{def}}$
  - $(snd\ (pair\ a\ b))$

    $\rightarrow b$

The *Pair* ADT

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  - $(pair\ a\ b) \rightarrow (\lambda xys.(s\ x\ y)\ a\ b) \rightarrow \lambda s.(s\ a\ b)$

- $fst \equiv_{\text{def}} \lambda p.(p\ T)$
  - $(fst\ (pair\ a\ b)) \rightarrow (\lambda p.(p\ T)\ \lambda s.(s\ a\ b)) \rightarrow (\lambda s.(s\ a\ b)\ T) \rightarrow (T\ a\ b) \rightarrow a$

- $snd \equiv_{\text{def}} \lambda p.(p\ F)$
  - $(snd\ (pair\ a\ b)) \rightarrow (\lambda p.(p\ F)\ \lambda s.(s\ a\ b)) \rightarrow (\lambda s.(s\ a\ b)\ F) \rightarrow (F\ a\ b) \rightarrow b$

The *List* ADT

Base constructors and operators

- Base constructors:

- Operators:

The *List* ADT

Base constructors and operators

- Base constructors:
  - $null : \rightarrow List$
- Operators:

The *List* ADT

Base constructors and operators

- Base constructors:
  - $null : \to List$
  - $cons : A \times List \to List$
- Operators:

The *List* ADT

Base constructors and operators

- Base constructors:
  - $null : \to List$
  - $cons : A \times List \to List$
- Operators:
  - $car : List \setminus \{null\} \to A$

The *List* ADT

Base constructors and operators

- Base constructors:
  - $null : \to List$
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- Operators:
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- Base constructors:
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- Operators:
  - $car : List \setminus \{null\} \to A$
  - $cdr : List \setminus \{null\} \to List$
  - $null? : List \to Bool$

The *List* ADT

Base constructors and operators

- Base constructors:
  - $null : \to List$
  - $cons : A \times List \to List$

- Operators:
  - $car : List \setminus \{null\} \to A$
  - $cdr : List \setminus \{null\} \to List$
  - $null? : List \to Bool$
  - $append : List^2 \to List$

The *List* ADT

Axioms

- *car*
- *cdr*
- *null*?
- *append*

The *List* ADT

Axioms

- *car*
  - $(car\ (cons\ e\ L)) = e$
- *cdr*
- *null*?
- *append*

The *List* ADT

Axioms

- *car*
  - $(car\ \ (cons\ \ e\ \ L)) = e$
- *cdr*
  - $(cdr\ \ (cons\ \ e\ \ L)) = L$
- *null*?
- *append*

The *List* ADT

Axioms

- *car*
  - $(car\ \ (cons\ \ e\ \ L)) = e$
- *cdr*
  - $(cdr\ \ (cons\ \ e\ \ L)) = L$
- *null?*
  - $(null?\ \ null) = T$
- *append*

The *List* ADT

Axioms

- *car*
  - $(car\ (cons\ e\ L)) = e$
- *cdr*
  - $(cdr\ (cons\ e\ L)) = L$
- *null*?
  - $(null?\ null) = T$
  - $(null?\ (cons\ e\ L)) = F$
- *append*

The *List* ADT

Axioms

- *car*
  - $(car\ (cons\ e\ L)) = e$
- *cdr*
  - $(cdr\ (cons\ e\ L)) = L$
- *null?*
  - $(null?\ null) = T$
  - $(null?\ (cons\ e\ L)) = F$
- *append*
  - $(append\ null\ B) = B$

The *List* ADT

Axioms

- *car*
  - $(car\ (cons\ e\ L)) = e$
- *cdr*
  - $(cdr\ (cons\ e\ L)) = L$
- *null?*
  - $(null?\ null) = T$
  - $(null?\ (cons\ e\ L)) = F$
- *append*
  - $(append\ null\ B) = B$
  - $(append\ (cons\ e\ A)\ B) = (cons\ e\ (append\ A\ B))$

The *List* ADT

Implementation

- Intuition:
- *null* $\equiv_{\text{def}}$
- *cons* $\equiv_{\text{def}}$
- *car* $\equiv_{\text{def}}$
- *cdr* $\equiv_{\text{def}}$
- *null*? $\equiv_{\text{def}}$
  - (*null*? *null*) $\rightarrow T$
  - (*null*? (*cons* *e* *L*)) $\rightarrow F$
- *append* $\equiv_{\text{def}}$

The *List* ADT

Implementation

- Intuition: a list = a (head, tail) pair
- $null \equiv_{\text{def}}$
- $cons \equiv_{\text{def}}$
- $car \equiv_{\text{def}}$
- $cdr \equiv_{\text{def}}$
- $null? \equiv_{\text{def}}$
  - $(null?\ null) \rightarrow T$
  - $(null?\ (cons\ e\ L))$
    $\rightarrow F$
- $append \equiv_{\text{def}}$

The *List* ADT

Implementation

- Intuition: a list = a (head, tail) pair
- $null \equiv_{\text{def}} \lambda x.T$
- $cons \equiv_{\text{def}}$
- $car \equiv_{\text{def}}$
- $cdr \equiv_{\text{def}}$
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  - $(null?\ \ null)$ $\rightarrow T$
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- Intuition: a list = a (head, tail) pair
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  - $(null?\ null)$ $\rightarrow T$
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- Intuition: a list = a (head, tail) pair
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- Intuition: a list = a (head, tail) pair
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- $cons \equiv_{\text{def}} pair$
- $car \equiv_{\text{def}} fst$
- $cdr \equiv_{\text{def}} snd$
- $null? \equiv_{\text{def}} \lambda L.(L\ \lambda xy.F)$
  - $(null?\ \ null) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda x.T) \rightarrow (\lambda x.T\ \ \ldots) \rightarrow T$
  - $(null?\ (cons\ e\ L)) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda s.(s\ e\ L)) \rightarrow (\lambda s.(s\ e\ L)\ \lambda xy.F) \rightarrow (\lambda xy.F\ e\ L) \rightarrow F$
- $append \equiv_{\text{def}}$

The *List* ADT

Implementation

- Intuition: a list = a (head, tail) pair
- $null \equiv_{\text{def}} \lambda x.T$
- $cons \equiv_{\text{def}} pair$
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- $null? \equiv_{\text{def}} \lambda L.(L\ \lambda xy.F)$
  - $(null?\ null) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda x.T) \rightarrow (\lambda x.T\ \ldots) \rightarrow T$
  - $(null?\ (cons\ e\ L)) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda s.(s\ e\ L)) \rightarrow (\lambda s.(s\ e\ L)\ \lambda xy.F) \rightarrow (\lambda xy.F\ e\ L) \rightarrow F$
- $append \equiv_{\text{def}}$ $\lambda AB.(if\ (null?\ A)\ B\ (cons\ (car\ A)\ (append\ (cdr\ A)\ B)))$

The *List* ADT

Implementation

- Intuition: a list = a (head, tail) pair
- $null \equiv_{\text{def}} \lambda x.T$
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  - $(null?\ null) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda x.T) \rightarrow (\lambda x.T\ \ldots) \rightarrow T$
  - $(null?\ (cons\ e\ L)) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda s.(s\ e\ L)) \rightarrow (\lambda s.(s\ e\ L)\ \lambda xy.F) \rightarrow (\lambda xy.F\ e\ L) \rightarrow F$
- $append \equiv_{\text{def}}$ … no closed form
  $\lambda AB.(if\ (null?\ A)\ B\ (cons\ (car\ A)\ (append\ (cdr\ A)\ B)))$

The *Natural* ADT

Axioms

- *zero*?
  - $(zero?\ zero) = T$
  - $(zero?\ (succ\ n)) = F$
- *pred*
  - $(pred\ (succ\ n)) = n$
- *add*
  - $(add\ zero\ n) = n$
  - $(add\ (succ\ m)\ n) = (succ\ (add\ m\ n))$

The *Natural* ADT

Implementation

- Intuition:

- *zero* $\equiv_{\text{def}}$
- *succ* $\equiv_{\text{def}}$
- *zero*? $\equiv_{\text{def}}$
- *pred* $\equiv_{\text{def}}$
- *add* $\equiv_{\text{def}}$

The *Natural* ADT

Implementation

- Intuition: a number = a list having the number value as its length
- *zero* $\equiv_{\text{def}}$
- *succ* $\equiv_{\text{def}}$
- *zero*? $\equiv_{\text{def}}$
- *pred* $\equiv_{\text{def}}$
- *add* $\equiv_{\text{def}}$

The *Natural* ADT

Implementation

- Intuition: a number = a list having the number value as its length
- $zero \equiv_{\text{def}} null$
- $succ \equiv_{\text{def}}$
- $zero? \equiv_{\text{def}}$
- $pred \equiv_{\text{def}}$
- $add \equiv_{\text{def}}$

The *Natural* ADT

Implementation

- Intuition: a number = a list having the number value as its length
- $zero \equiv_{\text{def}} null$
- $succ \equiv_{\text{def}} \lambda n.(cons\ \ null\ \ n)$
- $zero? \equiv_{\text{def}}$
- $pred \equiv_{\text{def}}$
- $add \equiv_{\text{def}}$

The *Natural* ADT
Implementation

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- $zero \equiv_{\text{def}} null$
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- $add \equiv_{\text{def}}$

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- $succ \equiv_{\text{def}} \lambda n.(cons\ \ null\ \ n)$
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- $add \equiv_{\text{def}} append$

Contents

Functions

- Several possible definitions of the identity function:

Functions

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Functions

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  - ...

Functions

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  - $\ldots$
- Infinitely many textual representations for the same function

Functions

- Several possible definitions of the identity function:
  - $id(n) = n$
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  - $\ldots$
- Infinitely many textual representations for the same function
- Then… what is a function?

Functions

- Several possible definitions of the identity function:
  - $id(n) = n$
  - $id(n) = n + 1 - 1$
  - $id(n) = n + 2 - 2$
  - …
- Infinitely many textual representations for the same function
- Then… what is a function? A relation between inputs and outputs, independent of any textual representation e.g.,
  $id = \{(0,0),(1,1),(2,2),\ldots\}$

Perspectives on recursion

- Textual: a function that refers itself, using its name

Perspectives on recursion

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- Constructivist: recursive functions as values of an ADT, with specific ways of building them

Perspectives on recursion

- Textual: a function that refers itself, using its name
- Constructivist: recursive functions as values of an ADT, with specific ways of building them
- Semantic: the mathematical object designated by a recursive function

Implementing *length*

Problem

- Length of a list:

$$length \equiv_{\text{def}} \lambda L.(if\ (null?\ L)\ zero\ (succ\ (\underline{length}\ (cdr\ L))))$$

Implementing *length*

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- Length of a list:

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Implementing *length*

Problem

- Length of a list:

  $$\textcolor{red}{length} \equiv_{\text{def}} \lambda L.(if\ (null?\ L)\ zero\ (succ\ (\underline{\textcolor{red}{length}}\ (cdr\ L))))$$

- What do we replace the underlined area with, to avoid textual recursion?
- Rewrite the definition as a fixed-point equation

  $$Length \equiv_{\text{def}} \lambda \textcolor{red}{f} L.(if\ (null?\ L)\ zero\ (succ\ (\textcolor{red}{f}\ (cdr\ L))))$$

  $$(Length\ length) \rightarrow length$$

Implementing *length*

Problem

- Length of a list:

  $length \equiv_{def} \lambda L.(if\ (null?\ L)\ zero\ (succ\ (\underline{length}\ (cdr\ L))))$

- What do we replace the underlined area with, to avoid textual recursion?
- Rewrite the definition as a fixed-point equation

  $Length \equiv_{def} \lambda f\, L.(if\ (null?\ L)\ zero\ (succ\ (f\ (cdr\ L))))$

  $(Length\ length) \rightarrow length$

- How do we compute the fixed point? (see code archive)

Contents

Axiomatization benefits

- Disambiguation
- Proof of properties
- Implementation skeleton

Syntax

- Variable:

$$Var ::= \text{any symbol distinct from } \lambda, ., (,)$$

Syntax

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$$\begin{aligned} Expr ::= \; & Var \\ \mid \; & \lambda Var . Expr \\ \mid \; & (Expr \;\; Expr) \end{aligned}$$

Syntax

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$$\mathit{Var} ::= \text{ any symbol distinct from } \lambda, ., (,)$$

- Expression:

$$\begin{aligned} \mathit{Expr} ::= \; & \mathit{Var} \\ \mid \; & \lambda \mathit{Var}.\mathit{Expr} \\ \mid \; & (\mathit{Expr} \;\; \mathit{Expr}) \end{aligned}$$

- Value:

$$\mathit{Val} ::= \lambda \mathit{Var}.\mathit{Expr}$$

Evaluation rules

Rule name:

$$\frac{\textit{precondition}_1, \ldots, \textit{precondition}_n}{\textit{conclusion}}$$

Semantics for normal-order evaluation

- *Reduce*:

$$(\lambda x.e \; e') \rightarrow e_{[e'/x]}$$

Semantics for normal-order evaluation

Evaluation

- *Reduce*:

$$(\lambda x.e \;\; e') \rightarrow e_{[e'/x]}$$

- *Eval*:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} =$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} =$

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Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
- $\langle \lambda x.e \rangle_{[e'/x]} =$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
- $\langle \lambda x.e \rangle_{[e'/x]} = \lambda x.e$
- $\langle \lambda y.e \rangle_{[e'/x]} =$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
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Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
- $\langle \lambda x.e \rangle_{[e'/x]} = \lambda x.e$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda y.e_{[e'/x]}, \quad y \neq x$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
- $\langle \lambda x.e \rangle_{[e'/x]} = \lambda x.e$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda y.e_{[e'/x]}, \quad y \neq x \wedge y \notin FV(e')$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda z.e_{[z/y][e'/x]},$
  $y \neq x \wedge y \in FV(e') \wedge z \notin FV(e) \cup FV(e')$
- $(e' \;\; e'')_{[e/x]} =$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
- $\langle \lambda x.e \rangle_{[e'/x]} = \lambda x.e$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda y.e_{[e'/x]}, \quad y \neq x \land y \notin FV(e')$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda z.e_{[z/y][e'/x]},$
  $y \neq x \land y \in FV(e') \land z \notin FV(e) \cup FV(e')$
- $(e' \; e'')_{[e/x]} = (e'_{[e/x]} \; e''_{[e/x]})$

Semantics for normal-order evaluation

Free variables

- $FV(x) = \{x\}$
- $FV(\lambda x.e) = FV(e) \setminus \{x\}$
- $FV((e' \ \ e'')) = FV(e') \cup FV(e'')$

Semantics for normal-order evaluation

Example

Example 12.1 (Evaluation rules).

$$((\lambda x.\lambda y.y \ \ a) \ \ b)$$

$$\frac{\qquad}{\qquad} \quad (\qquad)$$

Semantics for normal-order evaluation

Example

Example 12.1 (Evaluation rules).

$$((\lambda x.\lambda y.y \;\; a) \;\; b)$$

$$\frac{(\lambda x.\lambda y.y \;\; a) \rightarrow \lambda y.y \quad (Reduce)}{} \quad (\qquad)$$

Semantics for normal-order evaluation

Example

Example 12.1 (Evaluation rules).

$$((\lambda x.\lambda y.y\ \ a)\ \ b)$$

$$\dfrac{(\lambda x.\lambda y.y\ \ a) \rightarrow \lambda y.y \quad (Reduce)}{((\lambda x.\lambda y.y\ \ a)\ \ b) \rightarrow (\lambda y.y\ \ b)} \quad (Eval)$$

Semantics for normal-order evaluation

Example

Example 12.1 (Evaluation rules).

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$$\dfrac{(\lambda x.\lambda y.y \ \ a) \rightarrow \lambda y.y \quad (Reduce)}{((\lambda x.\lambda y.y \ \ a) \ \ b) \rightarrow (\lambda y.y \ \ b)} \quad (Eval)$$

$$(\lambda y.y \ \ b) \rightarrow b \quad (Reduce)$$

Semantics for applicative-order evaluation

- *Reduce* ($v \in Val$):

$$(\lambda x.e \ v) \rightarrow e_{[v/x]}$$

Semantics for applicative-order evaluation

Evaluation

- *Reduce* ($v \in Val$):

$$(\lambda x.e \;\; v) \rightarrow e_{[v/x]}$$

- $Eval_1$:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

Semantics for applicative-order evaluation

Evaluation

- *Reduce* ($v \in Val$):

$$(\lambda x.e\ \ v) \to e_{[v/x]}$$

- *Eval*$_1$:

$$\frac{e \to e'}{(e\ \ e'') \to (e'\ \ e'')}$$

- *Eval*$_2$ ($e \notin Val$):

$$\frac{e \to e'}{(\lambda x.e''\ \ e) \to (\lambda x.e''\ \ e')}$$

Formal proof

Proposition 12.2 (Free and bound variables).

$$\forall e \in Expr \bullet BV(e) \cap FV(e) = \emptyset$$

Formal proof

Proposition 12.2 (Free and bound variables).

$$\forall e \in Expr \bullet BV(e) \cap FV(e) = \emptyset$$

Proof.
Structural induction, according to the different forms of λ-expressions (see the lecture notes). □

Summary

- Practical usage of the untyped lambda calculus, as a programming language
- Formal specifications, for different evaluation semantics

Part IV

Typed Lambda Calculus

Contents

Contents

Drawbacks of the absence of types

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Drawbacks of the absence of types

- Meaningless expressions e.g., $(car\ 3)$
- No canonical representation for the values of a given type e.g., both a tree and a set having the same representation
- Impossibility of translating certain expressions into certain typed languages e.g., $(x\ x)$, Ω, Fix
- Potential irreducibility of expressions — inconsistent representation of equivalent values

$$\lambda x.(Fix\ x) \to \lambda x.(x\ (Fix\ x)) \to \lambda x.(x\ (x\ (Fix\ x))) \to \ldots$$

Solution

- Restricted ways of constructing expressions, depending on the types of their parts

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- Restricted ways of constructing expressions, depending on the types of their parts
- Sacrificed expressivity in change for soundness

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Definition 13.1 (Progress).

A well-typed expression is either a value or is subject to at least one reduction step.

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The result obtained by reducing a well-typed expression is well-typed. Usually, the type is the same.

Definition 13.3 (Strong normalization).
The evaluation of a well-typed expression terminates.

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Base and simple types

Definition 14.1 (Base type).

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- $v \in \tau$: "v is a value of type τ"
- $e \in \tau \Rightarrow e : \tau$
- $e : \tau \not\Rightarrow e \in \tau$

Typed λ-expressions

Definition 14.3 (λ_t-expression).

- Base value: a base value $b \in \tau_b$ is a λ_t-expression.

Typed λ-expressions

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Typed λ-expressions

Definition 14.3 (λ_t-expression).

- Base value: a base value $b \in \tau_b$ is a λ_t-expression.
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- Function: if $x : \sigma$ is a typed variable and $e : \tau$ is a λ_t-expression, then $\lambda x : \sigma.e : \sigma \to \tau$ is a λ_t-expression, which stands for ....

Typed λ-expressions

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- Base value: a base value $b \in \tau_b$ is a λ_t-expression.
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- Function: if $x : \sigma$ is a typed variable and $e : \tau$ is a λ_t-expression, then $\lambda x : \sigma . e : \sigma \to \tau$ is a λ_t-expression, which stands for ....
- Application: if $f : \sigma \to \tau$ and $a : \sigma$ are λ_t-expressions, then $(f \ a) : \tau$ is a λ_t-expression, which stands for ....

Relation to untyped lambda calculus

Similarities

- β-reduction
- α-conversion
- normal forms
- Church-Rosser theorem

Relation to untyped lambda calculus

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Differences

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Relation to untyped lambda calculus

Similarities

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Differences

- $(x : \tau \;\; x : \tau)$ invalid
- some fixed-point combinators are invalid

Syntax

Expressions

- Variables:

$$Var \quad ::= \quad \ldots$$

Syntax
Expressions

- Variables:

$$Var \quad ::= \quad \ldots$$

- Expressions:

$$\begin{array}{lcl} Expr & ::= & Val \\ & | & Var \\ & | & (Expr\ \ Expr) \end{array}$$

Syntax

Expressions

- Variables:

$$Var \quad ::= \quad \ldots$$

- Expressions:

$$\begin{array}{lll} Expr & ::= & Val \\ & \mid & Var \\ & \mid & (Expr\ \ Expr) \end{array}$$

- Values:

$$\begin{array}{lll} Val & ::= & \color{red}{BaseVal} \\ & \mid & \lambda Var \color{red}{:} \color{red}{Type}.Expr \end{array}$$

Syntax
Types

- Types:

$$\begin{array}{lcl} \mathit{Type} & ::= & \mathit{BaseType} \\ & | & (\mathit{Type} \rightarrow \mathit{Type}) \end{array}$$

Syntax

Types

- Types:

$$\begin{array}{lcl} \mathit{Type} & ::= & \mathit{BaseType} \\ & \mid & (\mathit{Type} \rightarrow \mathit{Type}) \end{array}$$

- Typing contexts:

$$\begin{array}{lcl} \mathit{TypingContext} & ::= & \emptyset \\ & \mid & \mathit{TypingContext}, \mathit{Var} : \mathit{Type} \end{array}$$

Syntax
Types

- Types:

$$\begin{array}{rcl} \mathit{Type} & ::= & \mathit{BaseType} \\ & | & (\mathit{Type} \rightarrow \mathit{Type}) \end{array}$$

- Typing contexts:
  - include variable-type associations
    i.e., *typing hypotheses*

$$\begin{array}{rcl} \mathit{TypingContext} & ::= & \emptyset \\ & | & \mathit{TypingContext}, \mathit{Var} : \mathit{Type} \end{array}$$

Semantics for normal-order evaluation

- *Reduce*:

$$(\lambda x : \tau . e \;\; e') \rightarrow e_{[e'/x]}$$

Semantics for normal-order evaluation

Evaluation

- *Reduce*:

$$(\lambda x : \tau . e \;\; e') \rightarrow e_{[e'/x]}$$

- *Eval*:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

Semantics for normal-order evaluation

- *Reduce*:

$$(\lambda x : \tau . e \;\; e') \rightarrow e_{[e'/x]}$$

- *Eval*:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

The type annotations are ignored,
since typing precedes evaluation.

Semantics

Typing

- *TBaseVal*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

Semantics

Typing

- *TBaseVal*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

Semantics

Typing

- *TBaseVal*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : (\tau \rightarrow \tau')}$$

Semantics

Typing

- *TBaseVal*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : (\tau \to \tau')}$$

- *TApp*:

$$\frac{\Gamma \vdash e : (\tau' \to \tau) \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e\ e') : \tau}$$

Typing example

Example 14.4 (Typing).

$$\lambda x : \tau_1 . \lambda y : \tau_2 . x : (\tau_1 \rightarrow (\tau_2 \rightarrow \tau_1))$$

Blackboard!

Type systems

Definition 14.5 (Type system).

The set of rules and mechanisms used in a programming language to organize, build and handle the types accepted in the language.

Type systems

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The set of rules and mechanisms used in a programming language to organize, build and handle the types accepted in the language.

Definition 14.6 (Soundness).

The type system of a language is *sound* if any well-typed expression in the language has the progress and preservation properties.

Type systems

Definition 14.5 (Type system).

The set of rules and mechanisms used in a programming language to organize, build and handle the types accepted in the language.

Definition 14.6 (Soundness).

The type system of a language is *sound* if any well-typed expression in the language has the progress and preservation properties.

Proposition 14.7.

STLC is sound and possesses the strong normalization property.

Contents

Ways of extending STLC

1. Particular base types

Ways of extending STLC

1. Particular base types

2. n-ary type constructors, $n \geq 1$, which build simple types

The product type

Algebraic specification

- Base constructors i.e., canonical values:

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$
- Operators:

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$
- Operators:
  - $\mathit{fst} : \tau * \tau' \rightarrow \tau$

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$
- Operators:
  - *fst* : $\tau * \tau' \to \tau$
  - *snd* : $\tau * \tau' \to \tau'$

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$
- Operators:
  - $\mathit{fst} : \tau * \tau' \rightarrow \tau$
  - $\mathit{snd} : \tau * \tau' \rightarrow \tau'$
- Axioms ($e : \tau$, $e' : \tau'$):

The product type

Algebraic specification

- Base constructors i.e., canonical values:
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- Operators:
  - $\mathit{fst} : \tau * \tau' \to \tau$
  - $\mathit{snd} : \tau * \tau' \to \tau'$
- Axioms ($e : \tau$, $e' : \tau'$):
  - $(\mathit{fst}\ (e, e')) \to e$

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$
- Operators:
  - $fst : \tau * \tau' \to \tau$
  - $snd : \tau * \tau' \to \tau'$
- Axioms ($e : \tau$, $e' : \tau'$):
  - $(fst\ (e, e')) \to e$
  - $(snd\ (e, e')) \to e'$

The product type

Syntax

The product type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{fst}\ \ \mathit{Expr}) \\
& | & (\mathit{snd}\ \ \mathit{Expr}) \\
& | & (\mathit{Expr}, \mathit{Expr})
\end{array}
$$

The product type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{fst}\ \ \mathit{Expr}) \\
& | & (\mathit{snd}\ \ \mathit{Expr}) \\
& | & (\mathit{Expr}, \mathit{Expr})
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{ProductVal}
\end{array}
$$

The product type

Syntax

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{fst}\ \ \mathit{Expr}) \\
& | & (\mathit{snd}\ \ \mathit{Expr}) \\
& | & (\mathit{Expr}, \mathit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{ProductVal}
\end{array}
$$

$$
\mathit{ProductVal} \quad ::= \quad (\mathit{Val}, \mathit{Val})
$$

The product type

Syntax

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{fst}\ \ \mathit{Expr}) \\
& | & (\mathit{snd}\ \ \mathit{Expr}) \\
& | & (\mathit{Expr}, \mathit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{ProductVal}
\end{array}
$$

$$
\mathit{ProductVal} \quad ::= \quad (\mathit{Val}, \mathit{Val})
$$

$$
\begin{array}{rcl}
\mathit{Type} & ::= & \ldots \\
& | & (\mathit{Type} * \mathit{Type})
\end{array}
$$

The product type

Evaluation

- *EvalFst*:

$$(fst\ \ (e, e')) \rightarrow e$$

The product type

Evaluation

- *EvalFst*:

$$(fst\ \ (e, e')) \rightarrow e$$

- *EvalSnd*:

$$(snd\ \ (e, e')) \rightarrow e'$$

The product type

Evaluation

- *EvalFst*:

$$(fst\ \ (e, e')) \rightarrow e$$

- *EvalSnd*:

$$(snd\ \ (e, e')) \rightarrow e'$$

- *EvalFstApp*:

$$\frac{e \rightarrow e'}{(fst\ \ e) \rightarrow (fst\ \ e')}$$

The product type

Evaluation

- *EvalFst*:

$$(fst\ \ (e, e')) \to e$$

- *EvalSnd*:

$$(snd\ \ (e, e')) \to e'$$

- *EvalFstApp*:

$$\frac{e \to e'}{(fst\ \ e) \to (fst\ \ e')}$$

- *EvalSndApp*:

$$\frac{e \to e'}{(snd\ \ e) \to (snd\ \ e')}$$

The product type

Typing

- *TProduct*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e, e') : (\tau * \tau')}$$

The product type

Typing

- *TProduct*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e, e') : (\tau * \tau')}$$

- *TFst*:

$$\frac{\Gamma \vdash e : (\tau * \tau')}{\Gamma \vdash (\mathit{fst}\ e) : \tau}$$

The product type

Typing

- *TProduct*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e, e') : (\tau * \tau')}$$

- *TFst*:

$$\frac{\Gamma \vdash e : (\tau * \tau')}{\Gamma \vdash (\mathit{fst}\ e) : \tau}$$

- *TSnd*:

$$\frac{\Gamma \vdash e : (\tau * \tau')}{\Gamma \vdash (\mathit{snd}\ e) : \tau'}$$

The product type

Typing example

Example 15.1 (Typing).

$$\begin{aligned}\Gamma \vdash\ &\lambda x : ((\rho * \tau) \to \sigma).\lambda y : \rho.\lambda z : \tau.(x\ \ (y, z)) \\ &: ((\rho * \tau) \to \sigma) \to \rho \to \tau \to \sigma\end{aligned}$$

Blackboard!

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - *Bool* $::=$ *True* | *False*

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathit{Bool} ::= \mathit{True} \mid \mathit{False}$
- Operators:

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathit{Bool} ::= \mathit{True} \mid \mathit{False}$
- Operators:
  - $\mathit{not} : \mathit{Bool} \rightarrow \mathit{Bool}$

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathit{Bool} ::= \mathit{True} \mid \mathit{False}$
- Operators:
  - $\mathit{not} : \mathit{Bool} \rightarrow \mathit{Bool}$
  - $\mathit{and} : \mathit{Bool}^2 \rightarrow \mathit{Bool}$

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $Bool ::= True \mid False$
- Operators:
  - $not : Bool \rightarrow Bool$
  - $and : Bool^2 \rightarrow Bool$
  - $or : Bool^2 \rightarrow Bool$

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $Bool ::= True \mid False$
- Operators:
  - $not : Bool \to Bool$
  - $and : Bool^2 \to Bool$
  - $or : Bool^2 \to Bool$
  - $if : Bool \times \tau \times \tau \to \tau$

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $Bool ::= True \mid False$
- Operators:
  - $not : Bool \rightarrow Bool$
  - $and : Bool^2 \rightarrow Bool$
  - $or : Bool^2 \rightarrow Bool$
  - $if : Bool \times \tau \times \tau \rightarrow \tau$
- Axioms: see slide 81

The *Bool* type

Syntax

The *Bool* type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
 & \mid & (\mathit{if}\ \ \mathit{Expr}\ \ \mathit{Expr}\ \ \mathit{Expr})
\end{array}
$$

The *Bool* type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
 & | & (\mathit{if}\ \ \mathit{Expr}\ \ \mathit{Expr}\ \ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{BaseVal} & ::= & \ldots \\
 & | & \mathit{BoolVal}
\end{array}
$$

The *Bool* type

Syntax

$$
\begin{aligned}
\mathit{Expr} \quad &::= \quad \ldots \\
&\;\,| \quad (\mathit{if}\ \ \mathit{Expr}\ \ \mathit{Expr}\ \ \mathit{Expr})
\end{aligned}
$$

$$
\begin{aligned}
\mathit{BaseVal} \quad &::= \quad \ldots \\
&\;\,| \quad \mathit{BoolVal}
\end{aligned}
$$

$$
\mathit{BoolVal} \quad ::= \quad \mathit{True} \mid \mathit{False}
$$

The *Bool* type

Syntax

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{if}\ \ \mathit{Expr}\ \ \mathit{Expr}\ \ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{BoolVal}
\end{array}
$$

$$
\mathit{BoolVal} \quad ::= \quad \mathit{True} \mid \mathit{False}
$$

$$
\begin{array}{rcl}
\mathit{BaseType} & ::= & \ldots \\
& | & \mathit{Bool}
\end{array}
$$

The *Bool* type

Evaluation

- *EvalIfT*:

$$(\mathit{if}\ \ \mathit{True}\ \ e\ \ e') \rightarrow e$$

The *Bool* type

Evaluation

- *EvalIfT*:

$$(\mathit{if}\ \ \mathit{True}\ \ e\ \ e') \rightarrow e$$

- *EvalIfF*:

$$(\mathit{if}\ \ \mathit{False}\ \ e\ \ e') \rightarrow e'$$

The *Bool* type

Evaluation

- *EvalIfT*:

$$(if\ \ True\ \ e\ \ e') \rightarrow e$$

- *EvalIfF*:

$$(if\ \ False\ \ e\ \ e') \rightarrow e'$$

- *EvalIf*:

$$\frac{e \rightarrow e'}{(if\ \ e\ \ e_1\ \ e_2) \rightarrow (if\ \ e'\ \ e_1\ \ e_2)}$$

The *Bool* type
Typing

- *TTrue*:

$$\Gamma \vdash \mathit{True} : \mathit{Bool}$$

The *Bool* type
Typing

- *TTrue*:

$$\Gamma \vdash \mathit{True} : \mathit{Bool}$$

- *TFalse*:

$$\Gamma \vdash \mathit{False} : \mathit{Bool}$$

The *Bool* type

Typing

- *TTrue*:

$$\Gamma \vdash \mathit{True} : \mathit{Bool}$$

- *TFalse*:

$$\Gamma \vdash \mathit{False} : \mathit{Bool}$$

- *TIf*:

$$\frac{\Gamma \vdash e : \mathit{Bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash (\mathit{if}\ \ e\ \ e_1\ \ e_2) : \tau}$$

The *Bool* type

Top-level variable bindings

The *Bool* type

Top-level variable bindings

- $not \equiv \lambda x : Bool.(if\ \ x\ \ False\ \ True)$

The *Bool* type

Top-level variable bindings

- $not \equiv \lambda x : Bool.(if\;\; x\;\; False\;\; True)$
- $and \equiv \lambda x : Bool.\lambda y : Bool.(if\;\; x\;\; y\;\; False)$

The *Bool* type

Top-level variable bindings

- $not \equiv \lambda x : Bool.(if\ \ x\ \ False\ \ True)$
- $and \equiv \lambda x : Bool.\lambda y : Bool.(if\ \ x\ \ y\ \ False)$
- $or \equiv \lambda x : Bool.\lambda y : Bool.(if\ \ x\ \ True\ \ y)$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\mathit{succ}\ \mathbb{N})$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
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- Operators:

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\textit{succ}\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \rightarrow \mathbb{N}$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\textit{succ}\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \to \mathbb{N}$
  - $\textit{zero}? : \mathbb{N} \to \textit{Bool}$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (succ\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \to \mathbb{N}$
  - $zero? : \mathbb{N} \to Bool$
- Axioms ($m, n \in \mathbb{N}$):

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (succ\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \to \mathbb{N}$
  - $zero? : \mathbb{N} \to Bool$
- Axioms ($m, n \in \mathbb{N}$):
  - $(+\ 0\ n) = n$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\mathit{succ}\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \to \mathbb{N}$
  - $\mathit{zero}? : \mathbb{N} \to \mathit{Bool}$
- Axioms ($m, n \in \mathbb{N}$):
  - $(+\ 0\ n) = n$
  - $(+\ (\mathit{succ}\ m)\ n) = (\mathit{succ}\ (+\ m\ n))$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\mathit{succ}\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \to \mathbb{N}$
  - $\mathit{zero}? : \mathbb{N} \to \mathit{Bool}$
- Axioms ($m, n \in \mathbb{N}$):
  - $(+\ 0\ n) = n$
  - $(+\ (\mathit{succ}\ m)\ n) = (\mathit{succ}\ (+\ m\ n))$
  - $(\mathit{zero}?\ 0) = \mathit{True}$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\mathit{succ}\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \rightarrow \mathbb{N}$
  - $\mathit{zero?} : \mathbb{N} \rightarrow \mathit{Bool}$
- Axioms ($m, n \in \mathbb{N}$):
  - $(+\ 0\ n) = n$
  - $(+\ (\mathit{succ}\ m)\ n) = (\mathit{succ}\ (+\ m\ n))$
  - $(\mathit{zero?}\ 0) = \mathit{True}$
  - $(\mathit{zero?}\ (\mathit{succ}\ n)) = \mathit{False}$

The $\mathbb{N}$ type

Operator semantics

- How to avoid defining evaluation and typing rules for each operator of $\mathbb{N}$?

The $\mathbb{N}$ type

Operator semantics

- How to avoid defining evaluation and typing rules for each operator of $\mathbb{N}$?

- Introduce the primitive recursor for $\mathbb{N}$, $prec_{\mathbb{N}}$, which allows for defining any primitive recursive function on natural numbers

The $\mathbb{N}$ type
Operator semantics

- How to avoid defining evaluation and typing rules for each operator of $\mathbb{N}$?
- Introduce the primitive recursor for $\mathbb{N}$, $prec_{\mathbb{N}}$, which allows for defining any primitive recursive function on natural numbers
- Define the operators using the primitive recursor

The $\mathbb{N}$ type

Syntax

The $\mathbb{N}$ type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{succ}\ \ \mathit{Expr}) \\
& | & (\mathit{prec}_{\mathbb{N}}\ \ \mathit{Expr}\ \ \mathit{Expr}\ \ \mathit{Expr})
\end{array}
$$

The $\mathbb{N}$ type

Syntax

$$
\begin{array}{rcl}
\textit{Expr} & ::= & \dots \\
& | & (\textit{succ}\ \textit{Expr}) \\
& | & (\textit{prec}_{\mathbb{N}}\ \textit{Expr}\ \textit{Expr}\ \textit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\textit{BaseVal} & ::= & \dots \\
& | & \textit{NVal}
\end{array}
$$

The $\mathbb{N}$ type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
 & | & (\mathit{succ}\ \mathit{Expr}) \\
 & | & (\mathit{prec}_{\mathbb{N}}\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{BaseVal} & ::= & \ldots \\
 & | & \mathit{NVal}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{NVal} & ::= & 0 \\
 & | & (\mathit{succ}\ \mathit{NVal})
\end{array}
$$

The $\mathbb{N}$ type

Syntax

$$\begin{array}{lcl} \mathit{Expr} & ::= & \ldots \\ & \mid & (\mathit{succ}\ \mathit{Expr}) \\ & \mid & (\mathit{prec}_{\mathbb{N}}\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr}) \end{array}$$

$$\begin{array}{lcl} \mathit{BaseVal} & ::= & \ldots \\ & \mid & \mathit{NVal} \end{array}$$

$$\begin{array}{lcl} \mathit{NVal} & ::= & 0 \\ & \mid & (\mathit{succ}\ \mathit{NVal}) \end{array}$$

$$\begin{array}{lcl} \mathit{BaseType} & ::= & \ldots \\ & \mid & \mathbb{N} \end{array}$$

The $\mathbb{N}$ type

Evaluation

- *EvalSucc*:

$$\frac{e \rightarrow e'}{(succ\ \ e) \rightarrow (succ\ \ e')}$$

The $\mathbb{N}$ type

Evaluation

- *EvalSucc*:

$$\frac{e \to e'}{(succ\ \ e) \to (succ\ \ e')}$$

- *EvalPrec*$_{\mathbb{N}0}$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ 0) \to e_0$$

The $\mathbb{N}$ type

Evaluation

- *EvalSucc*:

$$\frac{e \rightarrow e'}{(succ\ \ e) \rightarrow (succ\ \ e')}$$

- $EvalPrec_{\mathbb{N}0}$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ 0) \rightarrow e_0$$

- $EvalPrec_{\mathbb{N}1}$ $(n \in \mathbb{N})$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ (succ\ \ n)) \rightarrow (f\ \ n\ \ (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ n))$$

The $\mathbb{N}$ type

Evaluation

- *EvalSucc*:

$$\frac{e \to e'}{(succ\ \ e) \to (succ\ \ e')}$$

- $EvalPrec_{\mathbb{N}0}$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ 0) \to e_0$$

- $EvalPrec_{\mathbb{N}1}$ $(n \in \mathbb{N})$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ (succ\ \ n)) \to (f\ \ n\ \ (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ n))$$

- $EvalPrec_{\mathbb{N}2}$:

$$\frac{e \to e'}{(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ e) \to (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ e')}$$

The $\mathbb{N}$ type
Typing

- *TZero*:

$$\Gamma \vdash 0 : \mathbb{N}$$

The $\mathbb{N}$ type

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$$\Gamma \vdash 0 : \mathbb{N}$$

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The $\mathbb{N}$ type

Typing

- *TZero*:

$$\Gamma \vdash 0 : \mathbb{N}$$

- *TSucc*:

$$\frac{\Gamma \vdash e : \mathbb{N}}{\Gamma \vdash (succ\ \ e) : \mathbb{N}}$$

- *TPrec*$_{\mathbb{N}}$:

$$\frac{\Gamma \vdash e_0 : \tau \qquad \Gamma \vdash f : \mathbb{N} \to \tau \to \tau \qquad \Gamma \vdash e : \mathbb{N}}{\Gamma \vdash (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ e) : \tau}$$

The $\mathbb{N}$ type

Top-level variable bindings

The $\mathbb{N}$ type

Top-level variable bindings

- $zero? \equiv \lambda n : \mathbb{N}.(prec_{\mathbb{N}}\ \ True\ \ \lambda x : \mathbb{N}.\lambda y : Bool.False\ \ n)$

The $\mathbb{N}$ type

Top-level variable bindings

- $zero? \equiv \lambda n : \mathbb{N}.(prec_{\mathbb{N}}\ True\ \lambda x : \mathbb{N}.\lambda y : Bool.False\ n)$
- $+ \equiv \lambda m : \mathbb{N}.\lambda n : \mathbb{N}.(prec_{\mathbb{N}}\ n\ \lambda x : \mathbb{N}.\lambda y : \mathbb{N}.(succ\ y)\ m)$

The (*List* τ) type

Algebraic specification

- Base constructors i.e., canonical values:

The (*List* τ) type

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- Base constructors i.e., canonical values:
  - $(\mathit{List}\ \tau) \quad ::= \quad []_{\tau} \quad | \quad (\mathit{cons}\ \tau\ (\mathit{List}\ \tau))$

The ($List\ \tau$) type

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The (*List* τ) type

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- Operators:
  - $\mathit{head} : (\mathit{List}\ \ \tau) \setminus \{[]\} \rightarrow \tau$

The $(\mathit{List}\ \tau)$ type

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  - $\mathit{head} : (\mathit{List}\ \tau) \setminus \{[]\} \rightarrow \tau$
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- Operators:
  - $\mathit{head} : (\mathit{List}\ \ \tau) \setminus \{[]\} \rightarrow \tau$
  - $\mathit{tail} : (\mathit{List}\ \ \tau) \setminus \{[]\} \rightarrow (\mathit{List}\ \ \tau)$
  - $\mathit{length} : (\mathit{List}\ \ \tau) \rightarrow \mathbb{N}$

The $(List\ \tau)$ type

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- Base constructors i.e., canonical values:
  - $(List\ \tau) \quad ::= \quad []_{\tau} \quad | \quad (cons\ \tau\ (List\ \tau))$
- Operators:
  - $head : (List\ \tau) \setminus \{[]\} \rightarrow \tau$
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  - $length : (List\ \tau) \rightarrow \mathbb{N}$
- Axioms ($h \in \tau$, $t \in (List\ \tau)$):

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  - $length : (List\ \ \tau) \rightarrow \mathbb{N}$
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  - $(head\ \ (cons\ \ h\ \ t)) = h$

The $(List\ \tau)$ type

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- Base constructors i.e., canonical values:
  - $(List\ \tau) \quad ::= \quad []_\tau \quad | \quad (cons\ \tau\ (List\ \tau))$
- Operators:
  - $head : (List\ \tau) \setminus \{[]\} \to \tau$
  - $tail : (List\ \tau) \setminus \{[]\} \to (List\ \tau)$
  - $length : (List\ \tau) \to \mathbb{N}$
- Axioms ($h \in \tau$, $t \in (List\ \tau)$):
  - $(head\ (cons\ h\ t)) = h$
  - $(tail\ (cons\ h\ t)) = t$

The (*List* τ) type

Algebraic specification

- Base constructors i.e., canonical values:
  - $(List\ \ \tau) \quad ::= \quad []_{\tau} \quad | \quad (cons\ \ \tau\ \ (List\ \ \tau))$
- Operators:
  - $head : (List\ \ \tau) \setminus \{[]\} \to \tau$
  - $tail : (List\ \ \tau) \setminus \{[]\} \to (List\ \ \tau)$
  - $length : (List\ \ \tau) \to \mathbb{N}$
- Axioms ($h \in \tau$, $t \in (List\ \ \tau)$):
  - $(head\ \ (cons\ \ h\ \ t)) = h$
  - $(tail\ \ (cons\ \ h\ \ t)) = t$
  - $(length\ \ []) = 0$

The $(List\ \tau)$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $(List\ \tau) \quad ::= \quad []_\tau \quad | \quad (cons\ \tau\ (List\ \tau))$
- Operators:
  - $head : (List\ \tau) \setminus \{[]\} \to \tau$
  - $tail : (List\ \tau) \setminus \{[]\} \to (List\ \tau)$
  - $length : (List\ \tau) \to \mathbb{N}$
- Axioms ($h \in \tau$, $t \in (List\ \tau)$):
  - $(head\ (cons\ h\ t)) = h$
  - $(tail\ (cons\ h\ t)) = t$
  - $(length\ []) = 0$
  - $(length\ (cons\ h\ t)) = (succ\ (length\ t))$

The ($\mathit{List}\ \tau$) type

Syntax

The ($List\ \tau$) type

Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
 & | & (\mathit{cons}\ \mathit{Expr}\ \mathit{Expr}) \\
 & | & (\mathit{prec}_L\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr})
\end{array}
$$

The $(List\ \tau)$ type

Syntax

$$
\begin{array}{lcl}
Expr & ::= & \ldots \\
 & | & (cons\ Expr\ Expr) \\
 & | & (prec_L\ Expr\ Expr\ Expr)
\end{array}
$$

$$
\begin{array}{lcl}
BaseVal & ::= & \ldots \\
 & | & ListVal
\end{array}
$$

The $(\mathit{List}\ \tau)$ type
Syntax

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{cons}\ \mathit{Expr}\ \mathit{Expr}) \\
& | & (\mathit{prec}_L\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{ListVal}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{ListVal} & ::= & [] \\
& | & (\mathit{cons}\ \mathit{Value}\ \mathit{ListVal})
\end{array}
$$

The $(\mathit{List}\ \tau)$ type

Syntax

$$
\begin{array}{lll}
\mathit{Expr} & ::= & \ldots \\
 & | & (\mathit{cons}\ \mathit{Expr}\ \mathit{Expr}) \\
 & | & (\mathit{prec}_L\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{lll}
\mathit{BaseVal} & ::= & \ldots \\
 & | & \mathit{ListVal}
\end{array}
$$

$$
\begin{array}{lll}
\mathit{ListVal} & ::= & [] \\
 & | & (\mathit{cons}\ \mathit{Value}\ \mathit{ListVal})
\end{array}
$$

$$
\begin{array}{lll}
\mathit{Type} & ::= & \ldots \\
 & | & (\mathit{List}\ \mathit{Type})
\end{array}
$$

The (*List* τ) type

Evaluation

- *EvalCons*:

$$\frac{e \rightarrow e'}{(cons\ \ e\ \ e'') \rightarrow (cons\ \ e'\ \ e'')}$$

The $(List\ \ \tau)$ type

Evaluation

- *EvalCons*:

$$\frac{e \rightarrow e'}{(cons\ \ e\ \ e'') \rightarrow (cons\ \ e'\ \ e'')}$$

- *EvalPrec$_{L0}$*:

$$(prec_L\ \ e_0\ \ f\ \ []) \rightarrow e_0$$

The $(List\ \tau)$ type

Evaluation

- *EvalCons*:

$$\frac{e \rightarrow e'}{(cons\ e\ e'') \rightarrow (cons\ e'\ e'')}$$

- *EvalPrec$_{L0}$*:

$$(prec_L\ e_0\ f\ []) \rightarrow e_0$$

- *EvalPrec$_{L1}$* $(v \in Value)$:

$$(prec_L\ e_0\ f\ (cons\ v\ e)) \rightarrow (f\ v\ e\ (prec_L\ e_0\ f\ e))$$

The $(List\ \ \tau)$ type

Evaluation

- *EvalCons*:

$$\frac{e \to e'}{(cons\ \ e\ \ e'') \to (cons\ \ e'\ \ e'')}$$

- $EvalPrec_{L0}$:

$$(prec_L\ \ e_0\ \ f\ \ []) \to e_0$$

- $EvalPrec_{L1}$ $(v \in Value)$:

$$(prec_L\ \ e_0\ \ f\ \ (cons\ \ v\ \ e)) \to (f\ \ v\ \ e\ \ (prec_L\ \ e_0\ \ f\ \ e))$$

- $EvalPrec_{L2}$:

$$\frac{e \to e'}{(prec_L\ \ e_0\ \ f\ \ e) \to (prec_L\ \ e_0\ \ f\ \ e')}$$

The (*List* τ) type
Typing

- *TEmpty*:

$$\Gamma \vdash []_\tau : (\mathit{List}\ \ \tau)$$

The $(List\ \tau)$ type

Typing

- *TEmpty*:

$$\Gamma \vdash []_\tau : (List\ \tau)$$

- *TCons*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : (List\ \tau)}{\Gamma \vdash (cons\ e\ e') : (List\ \tau)}$$

The $(List\ \ \tau)$ type

Typing

- *TEmpty*:

$$\Gamma \vdash []_\tau : (List\ \ \tau)$$

- *TCons*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : (List\ \ \tau)}{\Gamma \vdash (cons\ \ e\ \ e') : (List\ \ \tau)}$$

- *TPrec$_L$*:

$$\frac{\Gamma \vdash e_0 : \tau' \quad \Gamma \vdash f : \tau \rightarrow (List\ \ \tau) \rightarrow \tau' \rightarrow \tau' \quad \Gamma \vdash e : (List\ \ \tau)}{\Gamma \vdash (prec_L\ \ e_0\ \ f\ \ e) : \tau'}$$

The (*List* τ) type

Top-level variable bindings

The $(List\ \ \tau)$ type

Top-level variable bindings

- $empty? \equiv \lambda l : (List\ \ \tau).(prec_L\ \ True\ \ f\ \ l)$,
  $f \equiv \lambda h : \tau.\lambda t : (List\ \ \tau).\lambda r : Bool.False$

The $(\mathit{List}\ \tau)$ type

Top-level variable bindings

- $\mathit{empty?} \equiv \lambda l : (\mathit{List}\ \tau).(\mathit{prec}_L\ \mathit{True}\ f\ l)$,
  $f \equiv \lambda h : \tau.\lambda t : (\mathit{List}\ \tau).\lambda r : \mathit{Bool}.\mathit{False}$

- $\mathit{length} \equiv \lambda l : (\mathit{List}\ \tau).(\mathit{prec}_L\ 0\ f\ l)$,
  $f \equiv \lambda h : \tau.\lambda t : (\mathit{List}\ \tau).\lambda r : \mathbb{N}.(\mathit{succ}\ r)$

General recursion

- Primitive recursion

General recursion

- Primitive recursion
  - induces *strong normalization*

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  - induces *strong normalization*
  - insufficient for capturing effectively computable functions

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- Introduce the operator *fix* i.e., a fixed-point combinator

General recursion

- Primitive recursion
  - induces *strong normalization*
  - insufficient for capturing effectively computable functions
- Introduce the operator *fix* i.e., a fixed-point combinator
- Gain computation power at the expense of strong normalization

fix

Syntax

fix

Syntax

$$\begin{array}{lcl} \textit{Expr} & ::= & \ldots \\ & | & (\textit{fix}\ \ \textit{Expr}) \end{array}$$

fix
Evaluation

- *EvalFix*:

$$(\mathit{fix}\ \ \lambda x : \tau.e) \rightarrow e_{[(\mathit{fix}\ \ \lambda x:\tau.e)/x]} \qquad = (f\ \ (\mathit{fix}\ \ f))$$

fix
Evaluation

- *EvalFix*:

$$(fix\ \ \lambda x : \tau.e) \rightarrow e_{[(fix\ \ \lambda x:\tau.e)/x]} \qquad = (f\ \ (fix\ \ f))$$

- *EvalFix'*:

$$\frac{e \rightarrow e'}{(fix\ \ e) \rightarrow (fix\ \ e')}$$

fix

Typing

- *TFix*:

$$\frac{\Gamma \vdash e : (\tau \rightarrow \tau)}{\Gamma \vdash (\mathit{fix}\ \ e) : \tau}$$

Example 15.2 (The *remainder* function).

$$
\begin{aligned}
&\textit{remainder} = \lambda m:\mathbb{N}.\lambda n:\mathbb{N}.\\
&\qquad((\textit{fix}\ \ \lambda f:(\mathbb{N}\to\mathbb{N}).\lambda p:\mathbb{N}.\\
&\qquad\qquad(\textit{if}\ \ p<n\ \ \textit{then}\ \ p\ \ \textit{else}\ \ (f\ \ (p-n))))\ \ m)
\end{aligned}
$$

The evaluation of (*remainder* 3 0) does not terminate.

Monomorphism

- Within the types $(\tau * \tau')$ and $(\textit{List}\ \ \tau)$, τ and τ' designate specific types e.g., *Bool*, $\mathbb{N}$, $(\textit{List}\ \ \mathbb{N})$, etc.

Monomorphism

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- Dedicated operators for each simple type

Monomorphism

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- $\mathit{fst}_{\mathbb{N},\mathit{Bool}}$, $\mathit{fst}_{\mathit{Bool},\mathbb{N}}$, $\ldots$

Monomorphism

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- $[]_{\mathbb{N}}$, $[]_{\mathit{Bool}}$, $\ldots$

Monomorphism

- Within the types $(\tau * \tau')$ and $(\mathit{List}\ \tau)$, τ and τ' designate specific types e.g., *Bool*, $\mathbb{N}$, $(\mathit{List}\ \mathbb{N})$, etc.
- Dedicated operators for each simple type
- $\mathit{fst}_{\mathbb{N},\mathit{Bool}}$, $\mathit{fst}_{\mathit{Bool},\mathbb{N}}$, $\ldots$
- $[]_{\mathbb{N}}$, $[]_{\mathit{Bool}}$, $\ldots$
- $\mathit{empty?}_{\mathbb{N}}$, $\mathit{empty?}_{\mathit{Bool}}$, $\ldots$

Contents

Idea

- Monomorphic identity function for type $\mathbb{N}$:

$$id_{\mathbb{N}} \equiv \lambda x : \mathbb{N}.x : (\mathbb{N} \to \mathbb{N})$$

Idea

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- Polymorphic identity function — type variables:

$$id \equiv \lambda X.\lambda x : X.x : \forall X.(X \to X)$$

Idea

- Monomorphic identity function for type $\mathbb{N}$:

$$id_{\mathbb{N}} \equiv \lambda x : \mathbb{N}.x : (\mathbb{N} \to \mathbb{N})$$

- Polymorphic identity function — type variables:

$$id \equiv \lambda X.\lambda x : X.x : \forall X.(X \to X)$$

- Type coercion prior to function application:

$$(id[\mathbb{N}]\ 5) \to (id_{\mathbb{N}}\ 5) \to 5$$

Syntax

- Program variables: stand for program values

$$\mathit{Var} \quad ::= \quad \ldots$$

Syntax

- Program variables: stand for program values

$$Var \quad ::= \quad \dots$$

- Type variables: stand for types

$$TypeVar \quad ::= \quad \dots$$

Syntax

- Expressions:

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \mathit{Value} \\
 & | & \mathit{Var} \\
 & | & (\mathit{Expr}\ \ \mathit{Expr}) \\
 & | & \mathit{Expr}[\mathit{Type}]
\end{array}
$$

Syntax

- Expressions:

$$\begin{array}{lcl} \mathit{Expr} & ::= & \mathit{Value} \\ & | & \mathit{Var} \\ & | & (\mathit{Expr}\ \ \mathit{Expr}) \\ & | & \mathit{Expr}[\mathit{Type}] \end{array}$$

- Values:

$$\begin{array}{lcl} \mathit{Value} & ::= & \mathit{BaseValue} \\ & | & \lambda\,\mathit{Var} : \mathit{Type}.\mathit{Expr} \\ & | & \lambda\,\mathit{TypeVar}.\mathit{Expr} \end{array}$$

Syntax

- Types:

$$
\begin{array}{lcl}
\mathit{Type} & ::= & \mathit{BaseType} \\
& | & \mathit{TypeVar} \\
& | & (\mathit{Type} \rightarrow \mathit{Type}) \\
& | & \forall \mathit{TypeVar}.\mathit{Type}
\end{array}
$$

Syntax

- Types:

$$
\begin{array}{rcl}
\mathit{Type} & ::= & \mathit{BaseType} \\
& | & \color{red}{\mathit{TypeVar}} \\
& | & (\mathit{Type} \to \mathit{Type}) \\
& | & \color{red}{\forall \mathit{TypeVar}.\mathit{Type}}
\end{array}
$$

- Typing contexts:

$$
\begin{array}{rcl}
\mathit{TypingContext} & ::= & \emptyset \\
& | & \mathit{TypingContext}, \mathit{Var} : \mathit{Type} \\
& | & \color{red}{\mathit{TypingContext}, \mathit{TypeVar}}
\end{array}
$$

Semantics

Evaluation

- *Reduce*$_1$:

$$(\lambda x : \tau . e \;\; e') \rightarrow e_{[e'/x]}$$

Semantics

Evaluation

- *Reduce*$_1$:

$$(\lambda x:\tau.e\ \ e') \to e_{[e'/x]}$$

- *Reduce*$_2$:

$$\lambda X.e[\tau] \to e_{[\tau/X]}$$

Semantics

Evaluation

- *Reduce*$_1$:

$$(\lambda x : \tau.e \;\; e') \rightarrow e_{[e'/x]}$$

- *Reduce*$_2$:

$$\lambda X.e[\tau] \rightarrow e_{[\tau/X]}$$

- *Eval*$_1$:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

Semantics

Evaluation

- *Reduce*$_1$:

$$(\lambda x : \tau.e \;\; e') \rightarrow e_{[e'/x]}$$

- *Reduce*$_2$:

$$\lambda X.e[\tau] \rightarrow e_{[\tau/X]}$$

- *Eval*$_1$:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

- *Eval*$_2$:

$$\frac{e \rightarrow e'}{e[\tau] \rightarrow e'[\tau]}$$

Semantics

Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

Semantics

Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

Semantics

Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*$_1$:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : (\tau \rightarrow \tau')}$$

Semantics

Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*$_1$:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : (\tau \rightarrow \tau')}$$

- *TApp*$_1$:

$$\frac{\Gamma \vdash e : (\tau' \rightarrow \tau) \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e\ e') : \tau}$$

Semantics

Typing

- *TAbs*$_2$ — polymorphic expressions have universal types:

$$\frac{\Gamma, X \vdash e : \tau}{\Gamma \vdash \lambda X.e : \forall X.\tau}$$

Semantics

Typing

- *TAbs*$_2$ — polymorphic expressions have universal types:

$$\frac{\Gamma, X \vdash e : \tau}{\Gamma \vdash \lambda X.e : \forall X.\tau}$$

- *TApp*$_2$:

$$\frac{\Gamma \vdash e : \forall X.\tau}{\Gamma \vdash e[\tau'] : \tau_{[\tau'/X]}}$$

Semantics

Substitution and free variables

- $\mathit{Expr}_{[\mathit{Expr} / \mathit{Var}]}$

Semantics

Substitution and free variables

- $Expr_{[Expr/Var]}$
- $Expr_{[Type/TypeVar]}$

Semantics

Substitution and free variables

- $\mathit{Expr}_{[\mathit{Expr}/\mathit{Var}]}$
- $\mathit{Expr}_{[\mathit{Type}/\mathit{TypeVar}]}$
- $\mathit{Type}_{[\mathit{Type}/\mathit{TypeVar}]}$

Semantics

Substitution and free variables

- $Expr_{[Expr/Var]}$
- $Expr_{[Type/TypeVar]}$
- $Type_{[Type/TypeVar]}$
- Free program variables

Semantics

Substitution and free variables

- $\mathit{Expr}_{[\mathit{Expr}/\mathit{Var}]}$
- $\mathit{Expr}_{[\mathit{Type}/\mathit{TypeVar}]}$
- $\mathit{Type}_{[\mathit{Type}/\mathit{TypeVar}]}$
- Free program variables
- Free type variables

Typing example

Example 16.1 (Typing).

$$\Gamma \vdash \lambda f : \forall X.(X \to X).\lambda Y.\lambda x : Y.(f[Y]\ x)$$
$$: (\forall X.(X \to X) \to \forall Y.(Y \to Y))$$

Typing example

Example 16.1 (Typing).

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$$: (\forall X.(X \to X) \to \forall Y.(Y \to Y))$$

Monomorphic function
with polymorphic argument and result!

Typing example

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Monomorphic function
with polymorphic argument and result!

Blackboard!

Examples of polymorphic expressions

Example 16.2 (Doubling a computation).

$$double \quad \equiv$$

Examples of polymorphic expressions

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$$double \quad \equiv \quad \lambda X.\lambda f : (X \rightarrow X).\lambda x : X.(f \ (f \ x))$$

Examples of polymorphic expressions

Example 16.2 (Doubling a computation).

$$\begin{array}{lll} \mathit{double} & \equiv & \lambda X.\lambda f:(X \to X).\lambda x:X.(f\ \ (f\ \ x)) \\ & : & \forall X.((X \to X) \to (X \to X)) \end{array}$$

Examples of polymorphic expressions

Example 16.2 (Doubling a computation).

$$\begin{array}{lcl} \mathit{double} & \equiv & \lambda X.\lambda f:(X \to X).\lambda x:X.(f\ \ (f\ \ x)) \\ & : & \forall X.((X \to X) \to (X \to X)) \end{array}$$

Example 16.3 (Quadrupling a computation).

$$\mathit{quadruple} \quad \equiv$$

Examples of polymorphic expressions

Example 16.2 (Doubling a computation).

$$\begin{aligned} \mathit{double} &\equiv \lambda X.\lambda f:(X \to X).\lambda x:X.(f\ \ (f\ \ x)) \\ &: \forall X.((X \to X) \to (X \to X)) \end{aligned}$$

Example 16.3 (Quadrupling a computation).

$$\mathit{quadruple} \equiv \lambda X.(\mathit{double}[X \to X]\ \ \mathit{double}[X])$$

Examples of polymorphic expressions

Example 16.2 (Doubling a computation).

$$
\begin{aligned}
\mathit{double} &\equiv \lambda X.\lambda f : (X \to X).\lambda x : X.(f\ \ (f\ \ x)) \\
&: \forall X.((X \to X) \to (X \to X))
\end{aligned}
$$

Example 16.3 (Quadrupling a computation).

$$
\begin{aligned}
\mathit{quadruple} &\equiv \lambda X.(\mathit{double}[X \to X]\ \ \mathit{double}[X]) \\
&: \forall X.((X \to X) \to (X \to X))
\end{aligned}
$$

Examples of polymorphic expressions

Example 16.4 (Reflexive computation).

$$\textit{reflexive} \quad \equiv$$

Examples of polymorphic expressions

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Examples of polymorphic expressions

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Example 16.5 (Fixed-point combinator).

$$\mathit{Fix} \equiv$$

Examples of polymorphic expressions

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Example 16.5 (Fixed-point combinator).

$$\mathit{Fix} \equiv \lambda X.\lambda f : (X \to X).(f\ (\mathit{Fix}[X]\ f))$$

Examples of polymorphic expressions

Example 16.4 (Reflexive computation).

$$\begin{aligned} \mathit{reflexive} &\equiv \lambda f : \forall X.(X \to X).(f[\forall X.(X \to X)]\ f) \\ &: (\forall X.(X \to X) \to \forall X.(X \to X)) \end{aligned}$$

Example 16.5 (Fixed-point combinator).

$$\begin{aligned} \mathit{Fix} &\equiv \lambda X.\lambda f : (X \to X).(f\ (\mathit{Fix}[X]\ f)) \\ &: \forall X.((X \to X) \to X) \end{aligned}$$

Contents

Motivation

Contents

Problem

- Polymorphic identity function, on objects of a type built using 1-ary type constructors e.g., *List*:

$$f \equiv \lambda C.\lambda X.\lambda x : (C\ \ X).x : \forall C.\forall X.((C\ \ X) \to (C\ \ X))$$

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- Monomorphic identity function for type $(List\ \ \mathbb{N})$:

$$f[List][\mathbb{N}] \to \lambda x : (List\ \ \mathbb{N}).x : ((List\ \ \mathbb{N}) \to (List\ \ \mathbb{N}))$$

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$$f[List][\mathbb{N}] \rightarrow \lambda x : (List\ \ \mathbb{N}).x : ((List\ \ \mathbb{N}) \rightarrow (List\ \ \mathbb{N}))$$

- How do we prevent erroneous situations e.g., $f[\mathbb{N}][\mathbb{N}]$, $f[List][List]$?

Solution

- Two categories of types: proper types, and type constructors i.e., $\lambda \mathit{TypeVar}.\mathit{Type}$

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- Type not only program variables, but also type variables

Solution

- Two categories of types: proper types, and type constructors i.e., $\lambda TypeVar.Type$
- Type not only program variables, but also type variables
- The type of a type: *kind*

Kinds
Notation

- The kind of a proper type: $*$

Kinds

Notation

- The kind of a proper type: $*$
- The kind of a 1-ary type constructor: $(* \Rightarrow *)$

Kinds

Notation

- The kind of a proper type: $*$
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- The kind of an *n*-ary type constructor, $n \geq 1$: $k_1 \Rightarrow k_2$

Kinds

Notation

- The kind of a proper type: $*$
- The kind of a 1-ary type constructor: $(* \Rightarrow *)$
- The kind of an *n*-ary type constructor, $n \geq 1$: $k_1 \Rightarrow k_2$
- The kind k of a type τ: $\tau :: k$

Example 18.1 (Kinds).

- $\mathbb{N}$

Kinds

Examples

Example 18.1 (Kinds).

- $\mathbb{N} :: *$
- *List*

Kinds
Examples

Example 18.1 (Kinds).

- $\mathbb{N} :: *$
- $\mathit{List} :: (* \Rightarrow *)$
- $f \equiv \lambda C :: (* \Rightarrow *).\lambda X :: *.\lambda x : (C\ X).x$

Example 18.1 (Kinds).

- $\mathbb{N} :: *$
- $List :: (* \Rightarrow *)$
- $f \equiv \lambda C :: (* \Rightarrow *).\lambda X :: *.\lambda x : (C\ \ X).x$
  $f : \forall C :: (* \Rightarrow *).\forall X :: *.((C\ \ X) \rightarrow (C\ \ X))$

Levels of expressions

Levels of expressions

Expressions

Levels of expressions

0

Expressions

Levels of expressions

0 $(0, \mathit{True})$

Expressions

Levels of expressions

0 $(0, True)$ $[][\mathbb{N}]$

Expressions

Levels of expressions

0 $(0, \mathit{True})$ $[][\mathbb{N}]$ $\lambda x : \mathbb{N}.x$

Expressions

Levels of expressions

$0 \qquad (0, \mathit{True}) \qquad [\,][\mathbb{N}] \qquad \lambda x : \mathbb{N}.x \qquad \lambda X :: *.\lambda x : X.x$

Expressions

Levels of expressions

Types

0 $(0, \mathit{True})$ $[][\mathbb{N}]$ $\lambda x : \mathbb{N}.x$ $\lambda X :: *.\lambda x : X.x$

Expressions

Levels of expressions

$\mathbb{N}$

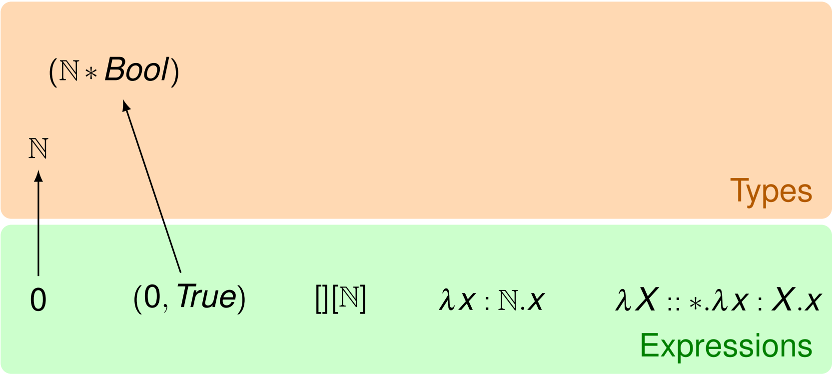
↑

Types

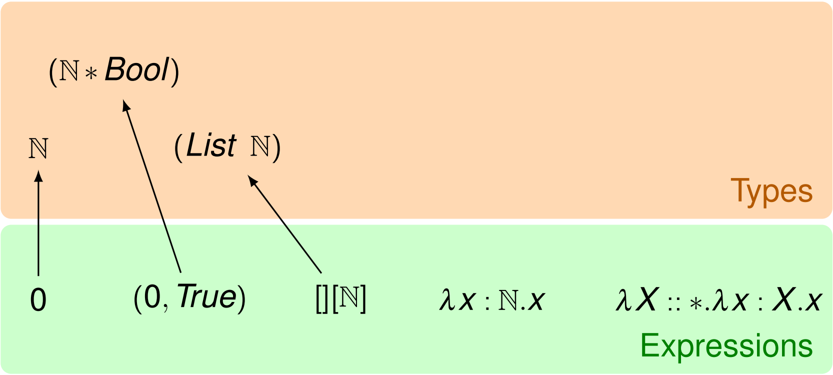
0 $\quad (0, \mathit{True}) \quad [\,][\mathbb{N}] \quad \lambda x : \mathbb{N}.x \quad \lambda X :: *.\lambda x : X.x$

Expressions

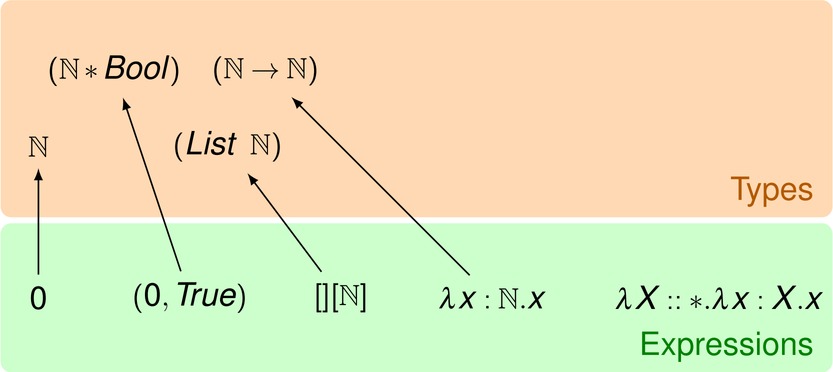
Levels of expressions



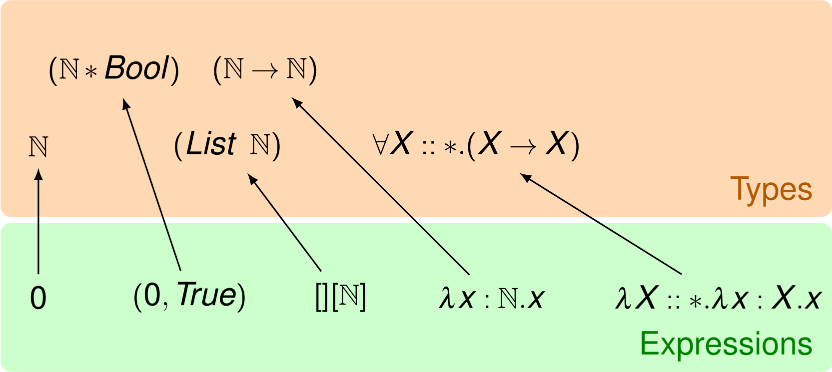
Levels of expressions



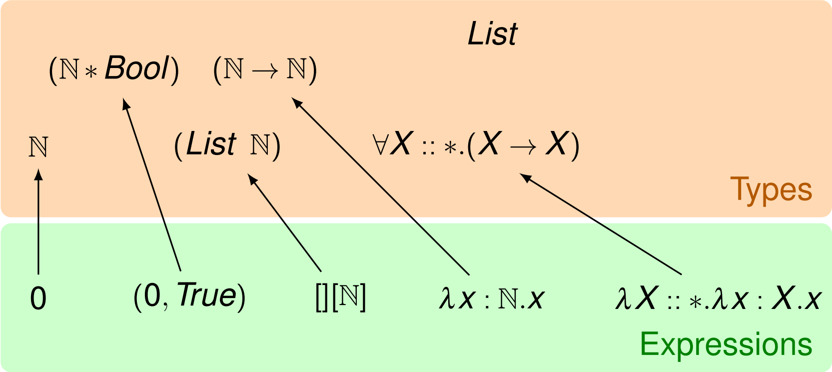
Levels of expressions



Levels of expressions



Levels of expressions



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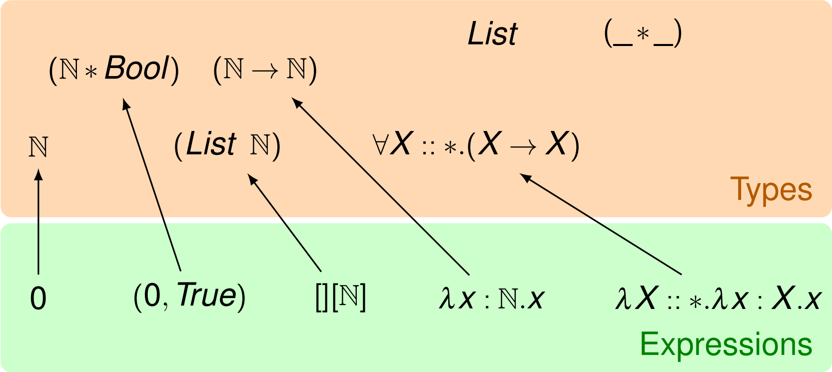
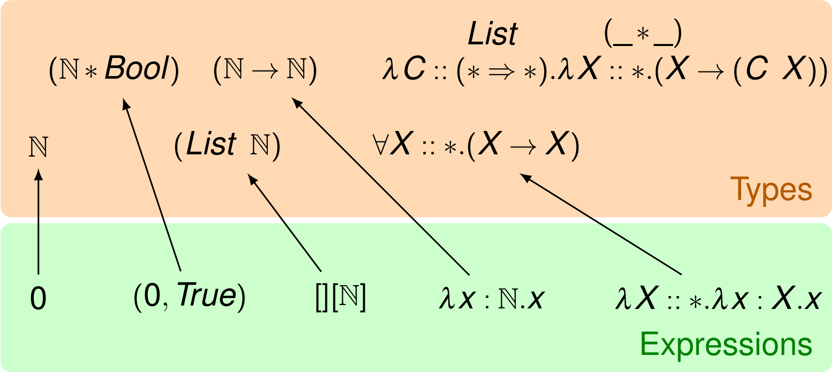
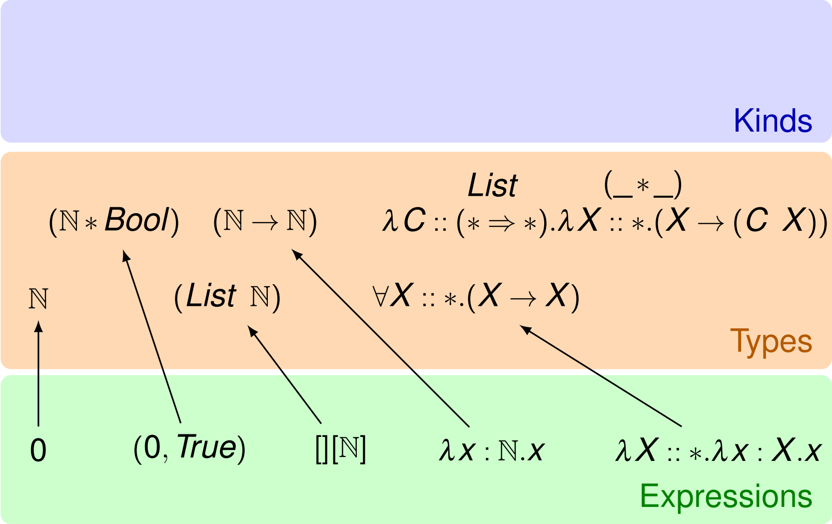


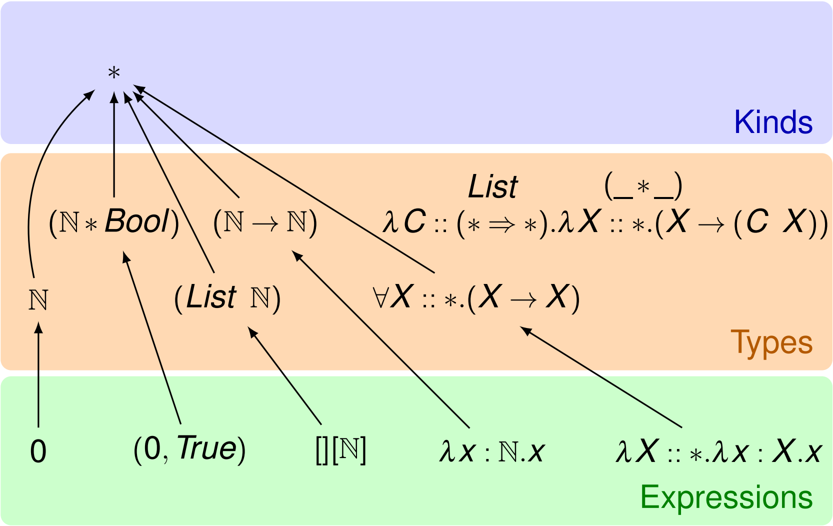
Levels of expressions



Levels of expressions



Levels of expressions



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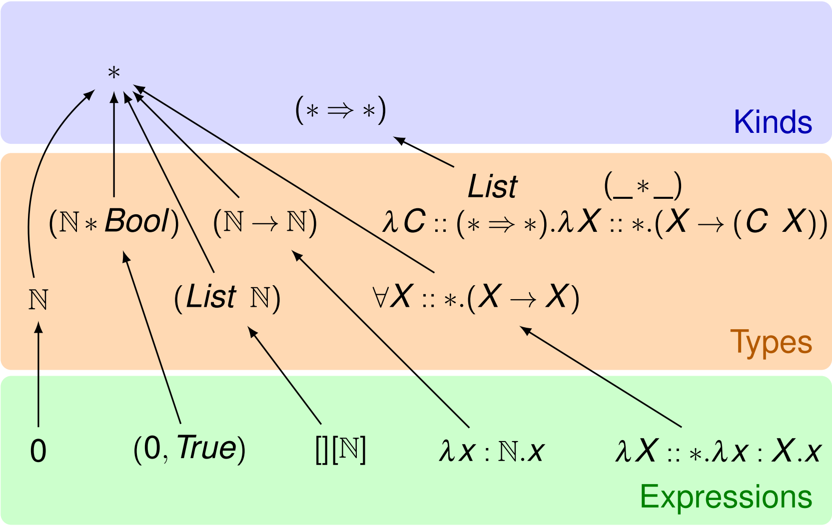

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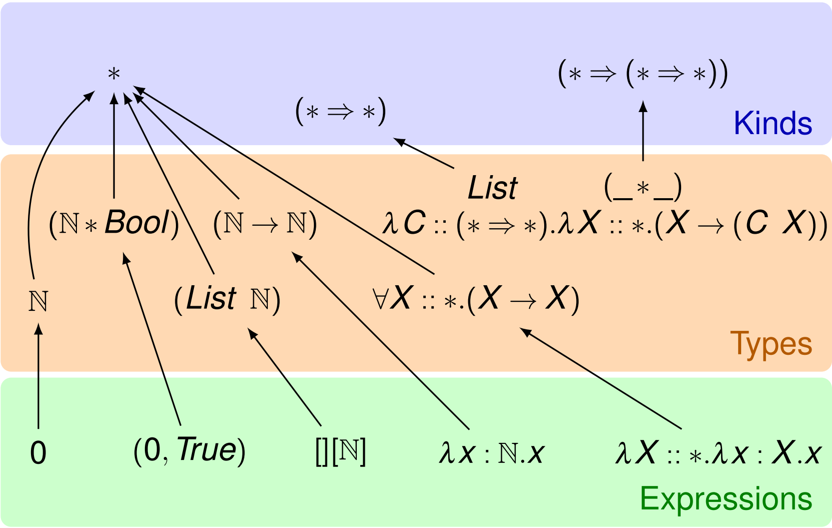

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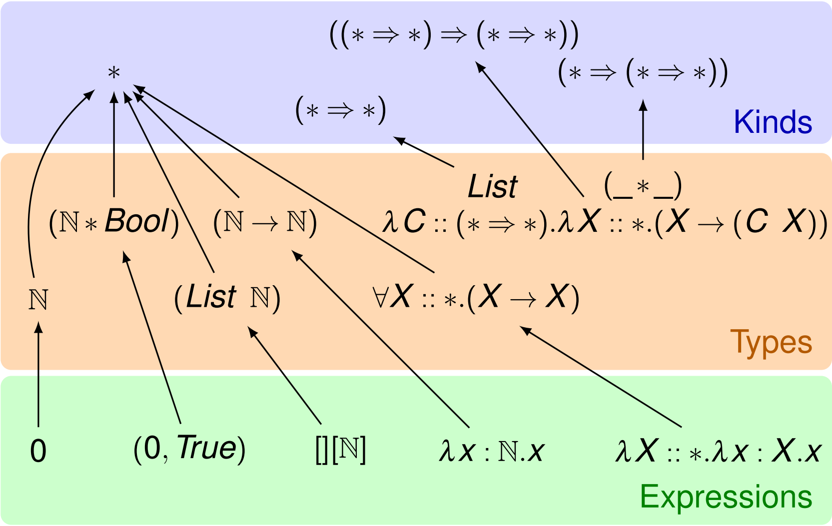

Type equivalence

- Two syntactically distinct types:

$$\tau_1 \equiv ((\mathit{List}\ \ \mathbb{N}) \rightarrow (\mathit{List}\ \ \mathbb{N}))$$
$$\tau_2 \equiv (\lambda X :: *.((\mathit{List}\ \ X) \rightarrow (\mathit{List}\ \ X))\ \ \mathbb{N})$$

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- Semantically, they denote the same type i.e., they are equivalent: $\tau_1 \equiv \tau_2$

Syntax

- Expressions:

$$
\begin{array}{lcl}
\mathit{Expr} & ::= & \mathit{Value} \\
 & | & \mathit{Var} \\
 & | & (\mathit{Expr}\ \ \mathit{Expr}) \\
 & | & \mathit{Expr}[\mathit{Type}]
\end{array}
$$

Syntax

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$$\begin{array}{lcl} \mathit{Expr} & ::= & \mathit{Value} \\ & | & \mathit{Var} \\ & | & (\mathit{Expr}\ \ \mathit{Expr}) \\ & | & \mathit{Expr}[\mathit{Type}] \end{array}$$

- Values:

$$\begin{array}{lcl} \mathit{Value} & ::= & \mathit{BaseValue} \\ & | & \lambda\,\mathit{Var} : \mathit{Type}.\mathit{Expr} \\ & | & \lambda\,\mathit{TypeVar} :: \mathit{Kind}.\mathit{Expr} \end{array}$$

Syntax

- Types:

$$\begin{array}{rcl} \mathit{Type} & ::= & \mathit{BaseType} \\ & \mid & \mathit{TypeVar} \\ & \mid & (\mathit{Type} \rightarrow \mathit{Type}) \\ & \mid & \forall \mathit{TypeVar} :: \mathit{Kind}.\mathit{Type} \\ & \mid & \lambda \mathit{TypeVar} :: \mathit{Kind}.\mathit{Type} \\ & \mid & (\mathit{Type}\ \ \mathit{Type}) \end{array}$$

Syntax

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\begin{array}{rcl}
\mathit{Type} & ::= & \mathit{BaseType} \\
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& | & \lambda \mathit{TypeVar} :: \mathit{Kind}.\mathit{Type} \\
& | & (\mathit{Type}\ \ \mathit{Type})
\end{array}
$$

- Typing contexts:

$$
\begin{array}{rcl}
\mathit{TypingContext} & ::= & \emptyset \\
& | & \mathit{TypingContext}, \mathit{Var} : \mathit{Type} \\
& | & \mathit{TypingContext}, \mathit{TypeVar} :: \mathit{Kind}
\end{array}
$$

Syntax

- Kinds:

$$\begin{array}{lcl} \mathit{Kind} & ::= & * \\ & | & (\mathit{Kind} \Rightarrow \mathit{Kind}) \end{array}$$

Semantics
Evaluation

- *Reduce*$_1$:

$$(\lambda x : \tau.e \;\; e') \rightarrow e_{[e'/x]}$$

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$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

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Semantics

Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

Semantics

Typing

- *TBaseValue*:

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- *TVar*:

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Semantics

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$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x.e : (\tau \rightarrow \tau')}$$

Semantics

Typing

- *TBaseValue*:
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$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x.e : (\tau \to \tau')}$$
- *TApp*$_1$:
$$\frac{\Gamma \vdash e : (\tau' \to \tau) \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e\ e') : \tau}$$

Semantics

Typing

- *TAbs*$_2$:

$$\frac{\Gamma, X :: K \vdash e : \tau}{\Gamma \vdash \lambda X :: K . e : \forall X :: K . \tau}$$

Semantics

Typing

- *TAbs*$_2$:

$$\frac{\Gamma, X :: K \vdash e : \tau}{\Gamma \vdash \lambda X :: K.e : \forall X :: K.\tau}$$

- *TApp*$_2$:

$$\frac{\Gamma \vdash e : \forall X :: K.\tau \qquad \Gamma \vdash \tau' :: K}{\Gamma \vdash e[\tau'] : \tau_{[\tau'/X]}}$$

Semantics

Kinding

- *KBaseType*:

$$\Gamma \vdash \tau_b :: *$$

Semantics

Kinding

- *KBaseType*:

$$\Gamma \vdash \tau_b :: *$$

- *KTypeVar*:

$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K}$$

Semantics

Kinding

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- *KTypeVar*:

$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K}$$

- *KTypeAbs*:

$$\frac{\Gamma, X :: K \vdash \tau :: K'}{\Gamma \vdash \lambda X :: K.\tau :: (K \Rightarrow K')}$$

Semantics

Kinding

- *KBaseType*:

$$\Gamma \vdash \tau_b :: *$$

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- *KTypeApp*:

$$\frac{\Gamma \vdash \tau :: (K' \Rightarrow K) \qquad \Gamma \vdash \tau' :: K'}{\Gamma \vdash (\tau\ \tau') :: K}$$

Semantics
Kinding

- *KAbs*$_1$:

$$\frac{\Gamma \vdash \tau :: * \qquad \Gamma \vdash \tau' :: *}{\Gamma \vdash (\tau \rightarrow \tau') :: *}$$

Semantics
Kinding

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- *KAbs*$_2$:

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Semantics

Type equivalence

- *EqReflexivity*:

$$\tau \equiv \tau$$

Semantics

Type equivalence

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- *EqSymmetry*:

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Semantics

Type equivalence

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- *EqSymmetry*:

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Semantics

Type equivalence

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$$\tau \equiv \tau$$

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$$\frac{\tau \equiv \tau'}{\tau' \equiv \tau}$$

- *EqTransitivity*:

$$\frac{\tau \equiv \tau' \qquad \tau' \equiv \tau''}{\tau \equiv \tau''}$$

- *EqTypeReduce*:

$$(\lambda X :: K.\tau \;\; \tau') \equiv \tau_{[\tau'/X]}$$

Semantics

Type equivalence

- *EqTypeAbs*:

$$\frac{\tau \equiv \tau'}{\lambda X :: K.\tau \equiv \lambda X :: K.\tau'}$$

Semantics

Type equivalence

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$$\frac{\tau \equiv \tau'}{\lambda X :: K.\tau \equiv \lambda X :: K.\tau'}$$

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Semantics

Type equivalence

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Semantics

Type equivalence

- *EqTypeAbs*:

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- *EqTypeApp*:

$$\frac{\tau \equiv \tau' \qquad \sigma \equiv \sigma'}{(\tau\ \ \sigma) \equiv (\tau'\ \ \sigma')}$$

- *EqAbs*$_1$:

$$\frac{\tau \equiv \tau' \qquad \sigma \equiv \sigma'}{(\tau \to \sigma) \equiv (\tau' \to \sigma')}$$

- *EqAbs*$_2$:

$$\frac{\tau \equiv \tau'}{\forall X :: K.\tau \equiv \forall X :: K.\tau'}$$

Semantics

Type equivalence

- *TypeEquivalence*:

$$\frac{\Gamma \vdash e : \tau \qquad \tau \equiv \tau'}{\Gamma \vdash e : \tau'}$$

Kinding example

Example 18.2 (Kinding).

$$\forall X :: *.(X \rightarrow ((List\ \ X) \rightarrow (Tree\ \ X)))$$

Kinding example

Example 18.2 (Kinding).

$$\forall X :: *.(X \to ((List\ \ X) \to (Tree\ \ X))) :: *$$

Blackboard!

Part V

Constructive Type Theory

Contents

Contents

Classical logic

- Example: prove $\exists x . P(x)$

Classical logic

- Example: prove $\exists x. P(x)$
- Perhaps, proof by contradiction: assume $\neg\exists x. P(x)$ and reach a contradiction

Classical logic

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- Assumption: $\exists x.P(x) \;\vee\; \neg\exists x.P(x)$ (law of excluded middle)

Classical logic

- Example: prove $\exists x. P(x)$
- Perhaps, proof by contradiction: assume $\neg\exists x. P(x)$ and reach a contradiction
- Assumption: $\exists x. P(x) \ \vee \ \neg\exists x. P(x)$ (law of excluded middle)
- Problem: possibly no actual evidence regarding either sentence i.e., some a s.t. either $P(a)$ or $\neg P(a)$ is true

Constructive logic

- Prove $\exists x.P(x)$ by computing an object a s.t. $P(a)$ is true

Constructive logic

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- However, not being able to compute a does not mean that $\exists x.P(x)$ is false

Constructive logic

- Prove $\exists x.P(x)$ by computing an object a s.t. $P(a)$ is true
- Not always possible
- However, not being able to compute a does not mean that $\exists x.P(x)$ is false
- Law of excluded middle — not an axiom in constructive logic

Constructive type theory

- Bridge between constructive logic and typed lambda calculus

Constructive type theory

- Bridge between constructive logic and typed lambda calculus

- Correspondences:

Constructive type theory

- Bridge between constructive logic and typed lambda calculus
- Correspondences:
  - sentence $\leftrightarrow$ type

Constructive type theory

- Bridge between constructive logic and typed lambda calculus
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  - sentence $\leftrightarrow$ type
  - logical connective $\leftrightarrow$ type constructor

Constructive type theory

- Bridge between constructive logic and typed lambda calculus
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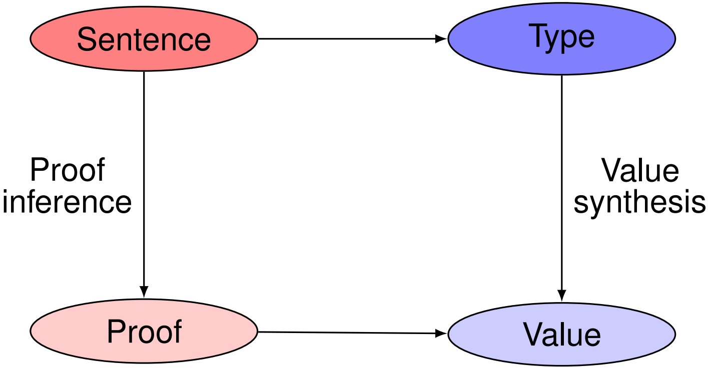
  - proof $\leftrightarrow$ function with that type

Constructive type theory

- Bridge between constructive logic and typed lambda calculus
- Correspondences:
  - sentence $\leftrightarrow$ type
  - logical connective $\leftrightarrow$ type constructor
  - proof $\leftrightarrow$ function with that type
- Application: synthesize a program by proving the sentence that corresponds to its specification

The Curry-Howard isomorphism



Contents

Two views

$$a : A$$

- Type-theoretic: "*a* is a value of type *A*"
- Logical: "*a* is a proof of sentence *A*"

Definitional rules

Rule	Logical view	Type-theoretic view

Definitional rules

Rule	Logical view	Type-theoretic view
Formation	How a connective relates two sentences	How a type constructor is used

Definitional rules

Rule	Logical view	Type-theoretic view
Formation	How a connective relates two sentences	How a type constructor is used
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Definitional rules

Rule	Logical view	Type-theoretic view
Formation	How a connective relates two sentences	How a type constructor is used
Introduction/ elimination	How a proof is derived	How a value is constructed
Computation	How a proof is simplified	How an expression is evaluated

Other logic-type correspondences

Logical view	Type-theoretic view

Other logic-type correspondences

Logical view	Type-theoretic view
Truth ($\top$)	One-element type, containing the trivial proof

Other logic-type correspondences

Logical view	Type-theoretic view
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Falsity ($\bot$)	No-element type, containing no proof

Other logic-type correspondences

Logical view	Type-theoretic view
Truth ($\top$)	One-element type, containing the trivial proof
Falsity ($\bot$)	No-element type, containing no proof
Proof by induction	Definition by recursion

Logical conjunction / product type constructor I

- Formation rule ($\wedge F$):

$$\frac{A \text{ is a sentence/ type} \qquad B \text{ is a sentence/ type}}{A \wedge B \text{ is a sentence/ type}}$$

- Introduction rule ($\wedge I$):

$$\frac{a : A \qquad b : B}{(a, b) : A \wedge B}$$

Logical conjunction / product type constructor II

- Elimination rules ($\wedge E_{1,2}$):

$$\frac{p : A \wedge B}{fst\ \ p : A} \qquad\qquad \frac{p : A \wedge B}{snd\ \ p : B}$$

- Computation rules:

$$fst\ \ (a, b) \to a$$
$$snd\ \ (a, b) \to b$$

Logical implication / function type constructor I

- Formation rule ($\Rightarrow F$):

$$\frac{A \text{ is a sentence/ type} \qquad B \text{ is a sentence/ type}}{A \Rightarrow B \text{ is a sentence/ type}}$$

- Introduction rule ($\Rightarrow I$)
(square brackets = discharged assumption):

$$\frac{\begin{array}{c}[x : A] \\ \vdots \\ b : B\end{array}}{\lambda x : A.b : A \Rightarrow B}$$

Logical implication / function type constructor II

- Elimination rule ($\Rightarrow E$):

$$\frac{a : A \qquad f : A \Rightarrow B}{(f \ \ a) : B}$$

- Computation rule:

$$(\lambda x : A.b \ \ a) \to b_{[a/x]}$$

Logical disjunction / sum type constructor I

- Formation rule ($\vee F$):

$$\frac{A \text{ is a sentence/ type} \qquad B \text{ is a sentence/ type}}{A \vee B \text{ is a sentence/ type}}$$

- Introduction rules ($\vee I_{1,2}$):

$$\frac{a : A}{inl\ \ a : A \vee B} \qquad\qquad \frac{b : B}{inr\ \ b : A \vee B}$$

Logical disjunction / sum type constructor II

- Elimination rule ($\vee E$):

$$\frac{p : A \vee B \qquad f : A \Rightarrow C \qquad g : B \Rightarrow C}{cases\ \ p\ \ f\ \ g : C}$$

- Computation rules:

$$cases\ \ (inl\ \ a)\ \ f\ \ g \rightarrow f\ \ a$$
$$cases\ \ (inr\ \ b)\ \ f\ \ g \rightarrow g\ \ b$$

Absurd sentence / empty type I

- Formation rule ($\perp F$):

$$\frac{}{\perp \text{ is a sentence/ type}}$$

- Introduction rule: none — there is no proof of the absurd sentence

Absurd sentence / empty type II

- Elimination rule ($\perp E$)
  (a proof of the absurd sentence can prove anything):

$$\frac{p : \perp}{abort_A\ p : A}$$

- Computation rule: none

Logical negation and equivalence

- Logical negation:

$$\neg A \;\equiv\; A \Rightarrow \bot$$

- Logical equivalence:

$$A \Leftrightarrow B \;\equiv\; (A \Rightarrow B) \wedge (B \Rightarrow A)$$

Example proofs

- $A \Rightarrow A$

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- $A \Rightarrow A$
- $A \Rightarrow \neg\neg A$ (converse?)

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Example proofs

- $A \Rightarrow A$
- $A \Rightarrow \neg\neg A$ (converse?)
- $((A \wedge B) \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$
- $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
- $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

Example proofs

- $A \Rightarrow A$
- $A \Rightarrow \neg\neg A$ (converse?)
- $((A \wedge B) \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$
- $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
- $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
- $(A \vee B) \Rightarrow \neg(\neg A \wedge \neg B)$

Universal quantification / generalized function type constructor I

- Formation rule ($\forall F$)
  (square brackets = discharged assumption):

$$\dfrac{A \text{ is a sentence/ type} \qquad \begin{array}{c}[x : A] \\ \vdots \\ B \text{ is a sentence/ type}\end{array}}{(\forall x : A).B \text{ is a sentence/ type}}$$

- Introduction rule ($\forall I$):

$$\dfrac{\begin{array}{c}[x : A] \\ \vdots \\ b : B\end{array}}{(\lambda x : A).b : (\forall x : A).B}$$

Universal quantification / generalized function type constructor II

- Elimination rule ($\forall E$):

$$\frac{a : A \qquad f : (\forall x : A).B}{(f \;\; a) : B_{[a/x]}}$$

- Computation rule:

$$((\lambda x : A).b \;\; a) \rightarrow b_{[a/x]}$$

Existential quantification / generalized product type constructor I

- Formation rule ($\exists F$)
  (square brackets = discharged assumption):

$$\frac{A \text{ is a sentence/ type} \qquad \begin{array}{c}[x:A] \\ \vdots \\ B \text{ is a sentence/ type}\end{array}}{(\exists x:A).B \text{ is a sentence/ type}}$$

- Introduction rule ($\exists I$):

$$\frac{a:A \qquad b:B_{[a/x]}}{(a,b):(\exists x:A).B}$$

Existential quantification / generalized product type constructor II

- Elimination rules ($\exists E_{1,2}$):

$$\frac{p : (\exists x : A).B}{Fst\ \ p : A} \qquad \frac{p : (\exists x : A).B}{Snd\ \ p : B_{[Fst\ \ p/x]}}$$

- Computation rules:

$$Fst\ \ (a, b) \rightarrow a$$
$$Snd\ \ (a, b) \rightarrow b$$

Example proofs

- $(\forall x : A).(B \Rightarrow C) \Rightarrow (\forall x : A).B \Rightarrow (\forall x : A).C$

Example proofs

- $(\forall x : A).(B \Rightarrow C) \Rightarrow (\forall x : A).B \Rightarrow (\forall x : A).C$
- $(\exists x : X).\neg P \Rightarrow \neg(\forall x : X).P$ (converse?)

Example proofs

- $(\forall x : A).(B \Rightarrow C) \Rightarrow (\forall x : A).B \Rightarrow (\forall x : A).C$
- $(\exists x : X).\neg P \Rightarrow \neg(\forall x : X).P$ (converse?)
- $(\exists y : Y).(\forall x : X).P \Rightarrow (\forall x : X).(\exists y : Y).P$ (converse?)